Example: $f(x) = \dfrac{x^2-1}{\sqrt{x^2+1}}$

$$f'(x) = \frac{2x\sqrt{x^2+1} - 2x\,\frac{1}{2}(x^2+1)^{-\frac{1}{2}}(x^2-1)}{\sqrt{x^2+1}^{\,2}}$$

$$= \frac{2x(x^2+1) - x(x^2-1)}{(x^2+1)^{3/2}} = \frac{2x^3+2x-x^3+x}{(x^2+1)^{3/2}}$$

$$= \frac{x^3+3x}{(x^2+1)^{3/2}} = \frac{x\overbrace{(3+x^2)}^{>0}}{\underbrace{(x^2+1)^{3/2}}_{>0}}$$

$f'(x) < 0$ if and only if $x<0 \Rightarrow f$ is decreasing

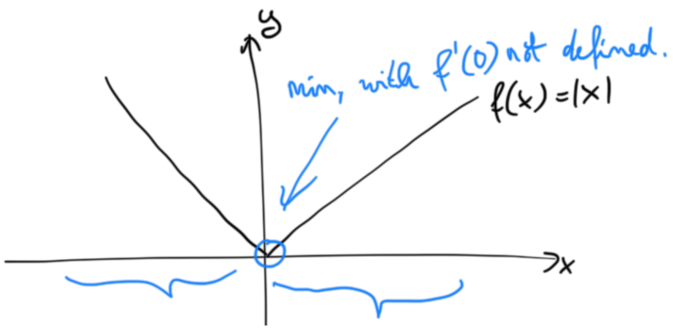
$f'(x) > 0$ " $x>0 \Rightarrow f$ is increasing

$\Rightarrow f$ has a global minimum at $x=0$

Note: at $x=0$, $f'(x)=0$. This is called a critical point.

Recall: If f has a local extreme value (min or max), and is differentiable at this point, then it has a critical point at this location.

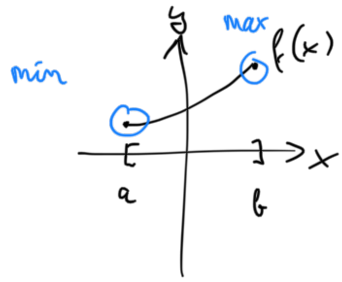
If f is not differentiable at this point, but is decreasing to the left, increasing to the right, it still has an extreme value even if $f'(x)$ is not defined. E.g.



$f'(x) = -1$, so f decreasing | $f'(x) = 1$, so f is increasing

Conclusion: Candidate points for extrema are

- critical points: $f'(x) = 0$
- Points where f' is not defined
- Endpoints of closed intervals of definition



Sufficient conditions for existence of an extreme value:

- If f' changes sign at the point in question
- $f''(x)$ exists and is non-zero:
  - → if $f''(x) > 0$, then (provided f'' exists and is continuous near x) f' is increasing hence, if $f'(x) = 0$, it must change sign from $-$ to $+$, so f has a min at x.
  - → if $f''(x) < 0$ ...... f has a maximum at x

Continue example from above:

$$f'(x) = \frac{x^3 + 3x}{(x^2+1)^{3/2}}$$

$$f''(x) = \frac{(3x^2+3)(x^2+1)^{\frac{3}{2}} - 2x\,\frac{3}{2}(x^2+1)^{\frac{1}{2}}(x^3+3x)}{(x^2+1)^3}$$

$$= \frac{(3x^2+3)(x^2+1) - 3x(x^3+3x)}{(x^2+1)^{5/2}}$$

$$= \frac{\cancel{3x^4} + 6x^2 + 3 - \cancel{3x^1} - 9x^2}{(x^2+1)^{5/2}}$$

$$= \frac{-3x^2 + 3}{\underbrace{(x^2+1)^{5/2}}_{>0}}$$

critical point at $x=0$:

$f''(x) > 0$ at $x=0$ $\Rightarrow$ f has a local minimum at $x=0$

(local because of the second derivative argument

But there is more information from f'':

If $f'' > 0$, then f' is increasing

$f'' < 0$, " f' is decreasing,

so if f'' changes sign at a point x, then the rate of change of the rate of change changes sign.

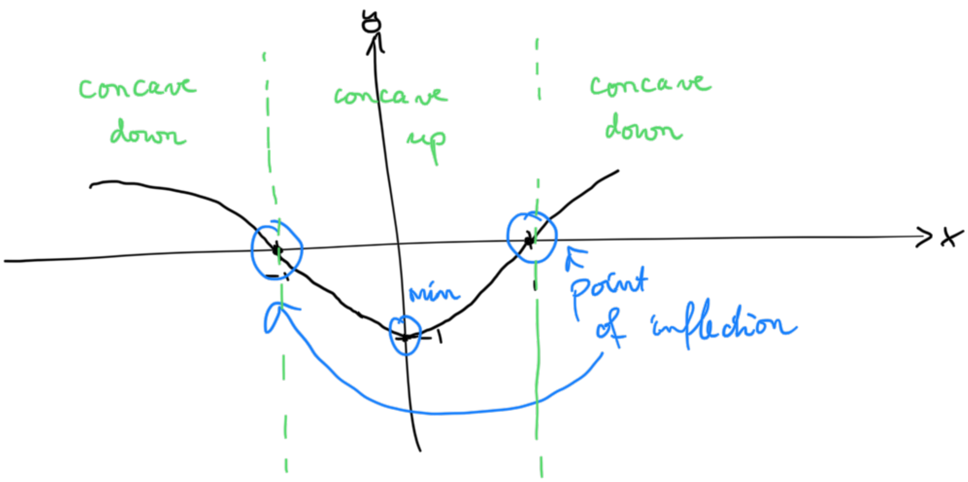
"acceleration"

Such a point is called point of inflection.

Here: $f''(x) = 3\frac{1-x^2}{(x^2+1)^{5/2}} = 3\frac{(1-x)(1+x)}{(x^2+1)^{5/2}}$

So at $x = -1$, f'' is changing sign from $-$ to $+$

$x = +1$, f'' is changing sign from $+$ to $-$



$f(x) = \frac{x^2-1}{\sqrt{x^2+1}}$

Def. when $f'' > 0$ on (a,b), we say that f is concave up $\rightarrow$ pos. curvature

(tangent line is below the graph of f)

$f'' < 0$ on (a,b), we say that f is concave down $\rightarrow$ neg. curvature

(tangent line is above graph of f)

Applications:

① Construct a rectangular container, square base

- material for base $5 \frac{€}{m^2}$
- material for sides/top $1 \frac{€}{m^2}$
- What is the largest possible volume for 72 € ?

$$V = b^2 h \qquad \text{(volume)}$$

$$C = 5\underbrace{b^2}_{\text{area of base}} + 1 \cdot (\underbrace{b^2}_{\text{area of top}} + \underbrace{4bh}_{\text{area of 4 sides}})$$

cost given: $5b^2 + b^2 + 4bh = 72 \implies \cancel{6}^{\,3} b^2 + \cancel{4}^{\,2} bh = \cancel{72}^{\,36}$

solve for h: $h = \dfrac{36 - 3b^2}{2b} = 18\dfrac{1}{b} - \dfrac{3}{2}b$

$$\Rightarrow V(b) = b^2\left(18\frac{1}{b} - \frac{3}{2}b\right) = 18b - \frac{3}{2}b^3$$

$$\Rightarrow V'(b) = 18 - \frac{9}{2}b^2 = 0 \implies 4 = b^2 \implies b = 2$$

(negative root does not make sense here)

$V''(b) = -9b < 0$ for $b > 0$, so we have a max. at $b = 2$

This corresponds to $h = 18\frac{1}{2} - \frac{3}{2} \cdot 2 = 9 - 3 = 6$

$$V = 2^2 \cdot 6 = 24$$

Alternative solution using implicit differentiation:

Suppose that both h and b are functions of some artificial parameter t.

$V = b^2 h$

Have two conditions: • $\frac{dV}{dt} = 0$ (necessary condition for max volume)

• $3b^2 + 2bh =$ fixed number

$$\frac{dV}{dt} = 2b\frac{db}{dt}h + b^2\frac{dh}{dt} = 0 \quad \Rightarrow \quad 2bh\frac{db}{dt} = -b^2\frac{dh}{dt} \quad (1)$$

$$3 \cdot 2b\frac{db}{dt} + 2\frac{db}{dt}h + 2b\frac{dh}{dt} = 0 \quad \Rightarrow \quad (6b+2h)\frac{db}{dt} = -2b\frac{dh}{dt} \quad (2)$$

Now divide (1) by (2):

$$\frac{2\cancel{b}h\cancel{\frac{db}{dt}}}{(6b+2h)\cancel{\frac{db}{dt}}} = \frac{\cancel{-}b^{\cancel{2}}\cancel{\frac{dh}{dt}}}{\cancel{-}2\cancel{b}\cancel{\frac{dh}{dt}}} \quad \Rightarrow \quad 4h = 6b + 2h$$

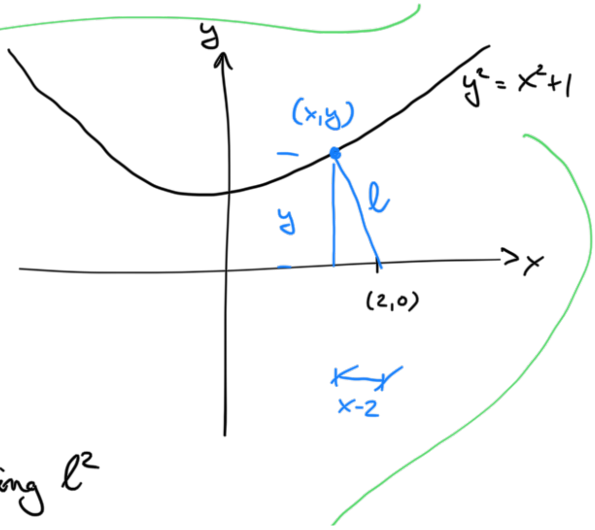
$$\Rightarrow \boxed{h = 3b}$$

Now get other quantities by simple algebra ....

This approach avoids solving before differentiating, so works more generally.

(2) What is the minimal distance of the point $(2,0)$ to the graph of $y^2 = ax^2+1$?

Distance l satisfies:



$$l^2 = (x-2)^2 + y^2$$

Here: $l^2 = (x-2)^2 + ax^2 + 1$

Note minimizing l is the same as minimizing l^2

$$\Rightarrow \quad 0 = \frac{d\ell^2}{dx} = 2(x-2) + a2x$$

$$= 4x - 4$$

$2(1+a)$

$\Rightarrow \quad x = 1$ $\qquad x = \frac{2}{1+a}$

$y = \sqrt{x^2+1}$

Why is this a minimum?

could look at second derivative, but here it's easier:

$\lim_{x \to \pm\infty} \ell^2 = \infty$ and there is only one critical point.

$\Rightarrow$ ℓ^2 cannot have a maximum, so critical point must correspond to a minimum.