Theorems of Differentiation

Theorem: $f : (a,b) \to \mathbb{R}$ diff'able at $x \in (a,b)$

$\Rightarrow f$ is cont. at x

f cont. at x_0:
$\lim\limits_{x \to x_0} f(x) = f(x_0)$
$\Leftrightarrow \lim\limits_{x \to x_0} f(x) - f(x_0) = 0$
$\Leftrightarrow \lim\limits_{h \to 0} f(x_0 + h) - f(x_0) = 0$

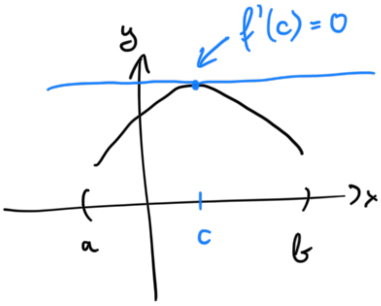
Proof:
$$\lim_{h \to 0} \big(f(x+h) - f(x) \big) = \lim_{h \to 0} h \cdot \frac{f(x+h) - f(x)}{h}$$
$$= \underbrace{\lim_{h \to 0} h}_{=0} \cdot \underbrace{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}_{= f'(x)}$$
since f diff'able at x $\square$

Note: f continuous in general does not imply f differentiable: $f(x) = |x|$ at $x = 0$

Theorem: $f : (a,b) \to \mathbb{R}$ diff'able

suppose f has a maximum (or a minimum) at some $c \in (a,b)$.

Then $f'(c) = 0$.

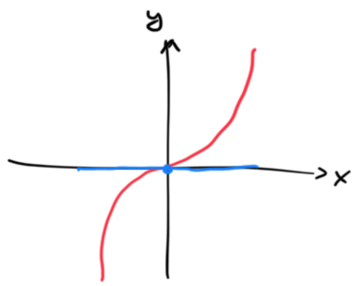


Note: $f'(x) = 0$ does not imply that f has a max (or min) at x.

E.g. $f(x) = x^3$

$f'(x) = 3x^2 \Rightarrow f'(0) = 0$

yet f is increasing and does not have a maximum at $x = 0$.



Proof: If the maximum of f in (a,b) is taken at c, then

$$f(c+h) - f(c) \leq 0$$

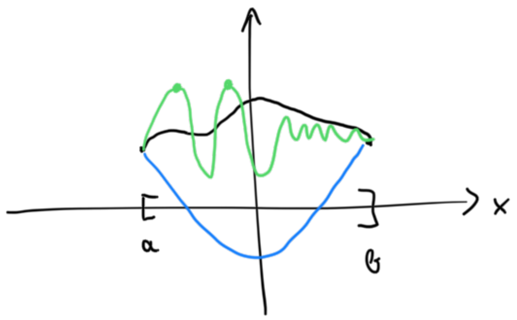
$h>0$:	$h<0$:
$\dfrac{f(c+h)-f(c)}{h} \leq 0$	$\dfrac{f(c+h)-f(c)}{h} \geq 0$
$\Rightarrow \underbrace{\lim_{h \searrow 0} \dfrac{f(c+h)-f(c)}{h}}_{f'(c)} \leq 0$	$\Rightarrow \underbrace{\lim_{h \nearrow 0} \dfrac{f(c+h)-f(c)}{h}}_{f'(c)} \geq 0$

$$f'(c) = 0$$

$\square$

Rolle's Theorem: $f: [a,b] \to \mathbb{R}$ cont., diff'able on (a,b), $f(a) = f(b)$

$$\Rightarrow \exists\, c \in (a,b) \text{ such that } f'(c) = 0$$



Informal statement: A "nice" function f has a root of f' between any two roots of f.

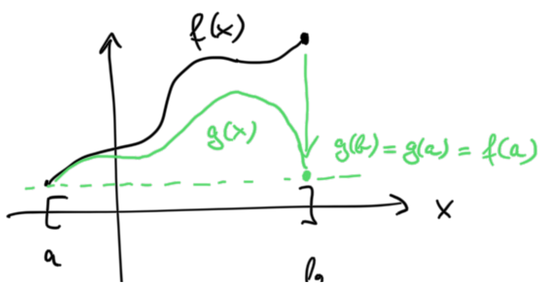
Proof: If f is a constant, then the statement is obvious because $f' = 0$ on (a,b).

Otherwise, as f is continuous, it must have its minimum and maximum on $[a,b]$ and at least one of them lies inside of (a,b).

Now previous result applies. $\square$

Q: What if $f(a) \neq f(b)$?

$$g(x) = f(x) - \frac{x-a}{b-a}\big(f(b) - f(a)\big)$$



$x=a: \quad 0$

$x=b: \quad 1$

$$g(a) = f(a)$$

$$g(b) = f(b) - \cancel{\frac{b-a}{b-a}}^{=1}\,(f(b)-f(a)) = f(a)$$

$$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

Rolle's theorem: There is $c \in (a,b)$ s.t. $g'(c) = 0$

instantaneous rate of change $\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$ average rate of change on the interval $[a,b]$

Informal statement: If f is "nice", the average rate of change over an interval equal the instantaneous rate of change somewhere within the interval.

The abstract statement is called <u>mean value theorem</u> (MVT):

$f : [a,b] \to \mathbb{R}$ cont., diff'able on (a,b)

$\Rightarrow$ there is $c \in (a,b)$ s.t. $\dfrac{f(b)-f(a)}{b-a} = f'(c)$.

<u>Example:</u> $f(x) = \ln x$, $f'(x) = \frac{1}{x}$

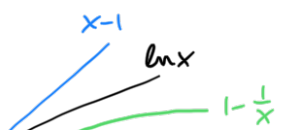
$$\frac{\ln x - \underbrace{\ln 1}_{=0}}{x-1} = \frac{1}{c} \quad \text{for } c \in (1,x) \quad (x>1)$$

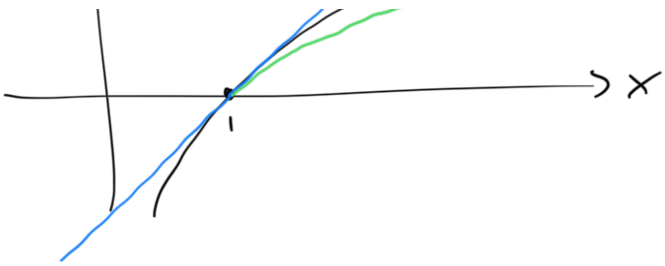
$$\Rightarrow \ln x = \frac{x-1}{c}$$

$c > 1 \Rightarrow \ln x < x-1$

$c < x \Rightarrow \ln x > \frac{x-1}{x}$

$\Big\}\ 1 - \frac{1}{x} < \ln x < x - 1$





Important consequence of MVT:

$f : (a,b) \to \mathbb{R}$ diff'able with $f'(x) \geq 0$ (≤ 0)

$\Rightarrow f$ is increasing (decreasing) on (a,b) I.e.: whenever $x_1 < x_2$, $x_1, x_2 \in (a,b)$ we have $f(x_1) \leq f(x_2)$ ($f(x_1) \geq f(x_2)$)

Proof: Suppose the contrary, i.e. there are $x_1 < x_2$ with $f(x_1) > f(x_2)$, $x_1, x_2 \in (a,b)$

$$\text{MVT:} \quad \underbrace{\frac{\overset{<0}{f(x_2) - f(x_1)}}{\underset{>0}{x_2 - x_1}}}_{<0} = \underbrace{f'(c)}_{\geq 0} \qquad \text{with } c \in (x_1, x_2)$$

contradiction

□

Examples: ① $f(x) = x^3 + x - 1$

$f'(x) = 3x^2 + 1 > 0$ for all $x \in \mathbb{R}$

$\Rightarrow f$ is increasing, so can have at most one real root.)

(Since $\lim_{x \to \infty} f(x) = \infty$, $\lim_{x \to -\infty} f(x) = -\infty$, it has exactly one root)

② $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2)$

$= 12x(x+1)(x-2)$

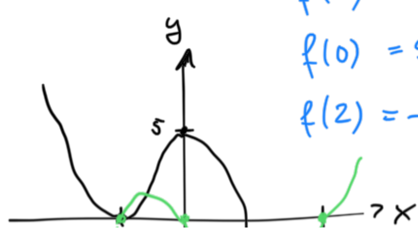
-1 is a root, divide out:

$(x^2 - x - 2) : (x+1) = x - 2$
$-(x^2 + x)$
$\quad -2x - 2$
$\quad -(-2x - 2)$
$\quad 0$

$f(-1) = 0$
$f(0) = 5$
$f(2) = -27$

$\Rightarrow$ $f'(x) < 0$, so f decreasing on $(-\infty, -1)$

$f'(x) > 0$, so f increasing on $(-1, 0)$



$f'(x) < 0$, so f decreasing on $(0, 2)$

$f'(x) > 0$, so f increasing on $(2, \infty)$