Derivative of the exponential function

$$(e^x)' = \lim_{h\to 0} \underbrace{\frac{e^{x+h}-e^x}{h}}_{\frac{e^x e^h - e^x}{h} = e^x \frac{e^h-1}{h}}$$

$$= e^x \underbrace{\lim_{h\to 0} \frac{e^h-1}{h}}_{=1} \quad \text{(Proof difficult, for Analysis I ...)}$$

So in short: $(e^x)' = e^x$

E.g.: $(e^{5x})' = 5e^{5x}$

Derivative of the inverse function

$f: (a,b) \to \mathbb{R}$ differentiable, invertible on its range, with inverse function $g = f^{-1}$

$g(f(x)) = x$ (definition of inverse)

Analysis I: f diff'able at x with $f'(x) \neq 0$
$\Rightarrow g$ diff'able at $y = f(x)$

Take derivative:

$$g'(f(x))\, f'(x) = 1$$

Let's solve for $g'(x)$:

$$g'(f(x)) = \frac{1}{f'(x)} \qquad \text{with } y = f(x): \qquad \boxed{g'(y) = \frac{1}{f'(x)} = \frac{1}{f'(g(y))}}$$

Examples: ① $f(x) = x^a$, $a > 0$ $\qquad f'(x) = a x^{a-1}$

$$g(y) = y^{\frac{1}{a}} = x$$

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{a x^{a-1}} = \frac{1}{a\left(y^{\frac{1}{a}}\right)^{a-1}} = \frac{1}{a} y^{-(1-\frac{1}{a})} = \frac{1}{a} y^{\frac{1}{a}-1}$$

② $f(x) = e^x \qquad f'(x) = e^x$

$g(y) = \ln y = x$

$$(\ln y)' = \frac{1}{e^x} = \frac{1}{e^{\ln y}} = \frac{1}{y}$$

Change notation: $\boxed{(\ln x)' = \frac{1}{x}}$

③ $f(x) = \tan x = \frac{\sin x}{\cos x} \qquad f'(x) = \frac{\cos x \cos x - \sin x(-\sin x)}{(\cos x)^2} = \frac{1}{\cos^2 x}$

$g(y) = \arctan y = x$

$$(\arctan y)' = \frac{1}{f'(x)} = \cos^2 x$$

Here: $y = \frac{\sin x}{\cos x} \Rightarrow y^2 = \frac{\sin^2 x}{\cos^2 x} = \frac{1-\cos^2 x}{\cos^2 x}$

$$\Rightarrow \cos^2 x \, y^2 = 1 - \cos^2 x$$

$$\Rightarrow \cos^2 x \, (1+y^2) = 1$$

$$\Rightarrow \cos^2 x = \frac{1}{1+y^2}$$

$$\Rightarrow \boxed{(\arctan y)' = \frac{1}{1+y^2}}$$

Implicit differentiation

Often have an equation that is difficult to write in terms of a function.

Eg.:

$$x^2 + y^2 = 1 \qquad (*) \qquad \text{(eqn. of unit circle)}$$

y is "dependent variable"

goal: $y'(x) = \frac{dy}{dx}$

x is "independent variable"

Naively: $y = \pm\sqrt{1-x^2}$

$$\frac{dy}{dx} = \pm \frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}} \underbrace{(-2x)}_{\text{"inner derivative"}} = \pm \frac{-x}{\sqrt{1-x^2}} = -\frac{x}{y}$$

Same via "implicit differentiation": Suppose $y = y(x)$ and differentiate $(*)$ w.r.t. x:

$$2x + 2y\frac{dy}{dx} = 0$$

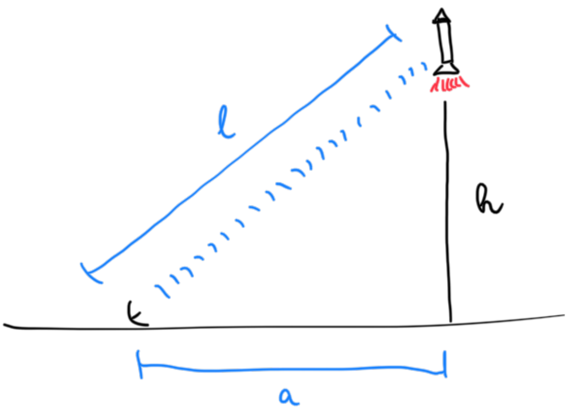
$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

Example: First, if $x(t)$ is distance traveled as a function of time, then

$$\overline{v} = \frac{x(t_2) - x(t_1)}{t_2 - t_1} = \frac{\text{distance traveled between } t_1 \text{ and } t_2}{\text{travel time}}$$

$$v(t) = \lim_{t_2 \to t} \frac{x(t_2) - x(t)}{t_2 - t} = \frac{dx}{dt} \quad \text{instantaneous velocity}$$

Here:



Radar can track $\ell(t)$ and therefore approximate $\ell'(t) = \frac{d\ell}{dt}$

Q: what is the vertical velocity $\frac{dh}{dt}$?

$$\ell^2 = a^2 + h^2$$

$$\Rightarrow 2\ell\frac{d\ell}{dt} = 0 + 2h\frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{\ell}{h}\frac{d\ell}{dt} = \frac{\ell}{\sqrt{\ell^2 - a^2}}\frac{d\ell}{dt}$$

measurable quantities

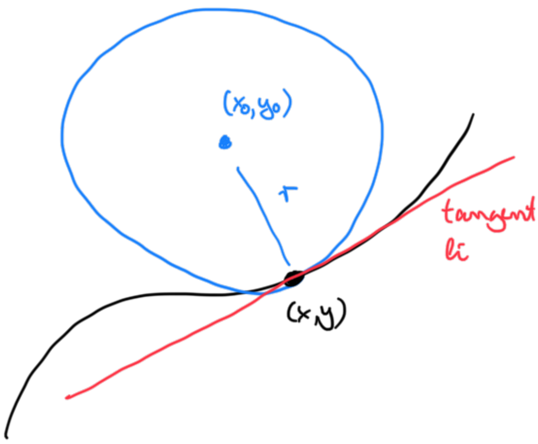
$\Rightarrow \quad dt \qquad h \qquad dt \qquad \sqrt{\ell^2 - a}$

E.g. $a = 3\ \text{km}$

$\ell = 5\ \text{km} \qquad \frac{d\ell}{dt} = 1000\ \frac{m}{s}$

$$\Rightarrow \frac{dh}{dt} = \frac{5\ \text{km}}{\underbrace{\sqrt{25\ \text{km}^2 - 9\ \text{km}^2}}_{\frac{5}{4}}} \quad 1000\ \frac{m}{s} = 1250\ \frac{m}{s}$$

Application to curvature of the graph of a function:



osculating circle:

- touches graph of f at point (x,y)
- it has the same first and second derivative at that point.

"osculating circle provides a second order local approximation to the graph of f, while the tangent line is only a first-order approximation".

Q: what is the radius r of the osculating circle?

① $(x-x_0)^2 + (y-y_0)^2 = r^2$ (general circle equation)

Differentiate in x:

② $2(x-x_0) + 2(y-y_0)\frac{dy}{dx} = 0 \Rightarrow x - x_0 + (y-y_0)\frac{dy}{dx} = 0$

One more derivative:

$\frac{d^2y}{dx^2} = y''(x) = (y'(x))'$

③ $1 + \frac{dy}{dx}\frac{dy}{dx} + (y-y_0)\frac{d^2y}{dx^2} = 0$

Osculating circle conditions:

circle graph	function f
y	$f(x)$
$\frac{dy}{dx}$	$f'(x)$
$\frac{d^2y}{dx^2}$	$f''(x)$

From ③: $(y-y_0)\, f''(x) = -1 - f'(x)^2 \quad \Rightarrow \quad y - y_0 = -\dfrac{1+f'(x)^2}{f''(x)}$

From ②: $x - x_0 = -(y-y_0)\, f'(x) = f'(x)\, \dfrac{1+f'(x)^2}{f''(x)}$

From ①: $r^2 = (x-x_0)^2 + (y-y_0)^2 = f'(x)^2 \left(\dfrac{1+f'(x)^2}{f''(x)}\right)^2 + \left(\dfrac{1+f'(x)^2}{f''(x)}\right)^2$

$$= \frac{\left(1+f'(x)^2\right)^3}{f''(x)^2}$$

Now define <u>curvature</u> $\boxed{K = \frac{1}{r} = \dfrac{f''(x)}{\left(1+f'(x)^2\right)^{3/2}}}$

"convex"

$f''(x) > 0$ say "f is concave up", i.e. osculating circle lies above graph of f

$f''(x) < 0$ say "f is concave down" or "concave", osculating circle lies below the graph.