$$\lim_{x\to 0} x \ln x = 0$$

$$\boxed{\lim_{x\to 0} x^{\alpha} \ln x = 0 \quad \text{for every } \alpha > 0}$$

$$\ln x \, e^{-x} \longrightarrow 0$$

$$\| $$

$$\underbrace{\frac{\ln x}{x}}_{\to 0} \quad \underbrace{x e^{-x}}_{\to 0} \quad \text{as } x \to \infty$$

When can we split up the limit of a product?

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1} \qquad\qquad \lim_{x\to 0} \frac{\sin^2 x}{x^2} = \left(\lim_{x\to 0} \frac{\sin x}{x}\right)^2$$

<u>Recall:</u> $f:(a,b) \to \mathbb{R}$ is differentiable at $x \in (a,b)$ if

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \quad \text{exists}$$

"derivative"

We say that <u>f is differentiable</u> if it is differentiable at every $x \in (a,b)$

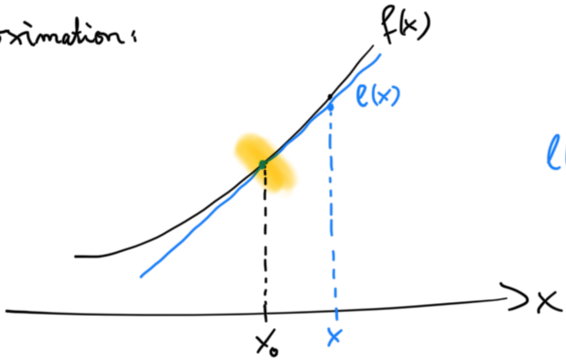
<u>Remark:</u> we can also write

$$f'(x) = \lim_{\xi \to x} \frac{f(\xi) - f(x)}{\xi - x}$$

alternative notation: $f'(x) = \frac{df}{dx} = \dot{f} = Df$

The derivative of f can be seen as the "slope of f at point x" or "instantaneous rate of change"

Tangent line approximation:



$$\ell(x) = f(x_0) + f'(x_0)(x - x_0)$$

Relative error of approximation by the tangent line:

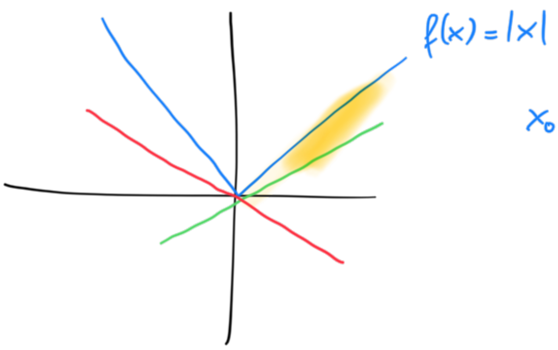
$$\frac{f(x) - \ell(x)}{x - x_0} = \frac{f(x) - \big(f(x_0) + f'(x_0)(x - x_0)\big)}{x - x_0}$$

$$= \underbrace{\frac{f(x) - f(x_0)}{x - x_0}}_{\longrightarrow f'(x_0) \text{ if } f \text{ is differentiable}} - f'(x_0)$$

$$\longrightarrow 0 \qquad \text{〃}$$

- For a differentiable function, the relative error of approximation is going to zero when approaching the point where the line is tangent.

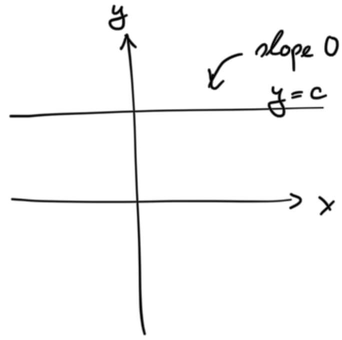
- A differentiable function is characterized by the fact that it can be approximated locally by a linear function.



$x_0 = 0$

Here, relative error of approximation does not vanish as $x \to 0$ for any of the many possible tangent lines.

Rules of differentiation:



$$(f+g)' = f' + g'$$

$$(fg)' = f'g + g'f \qquad \text{"Product rule"}$$

$c \in \mathbb{R}$ is a constant, $c' = 0$

$$(cf)' = cf'$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2} \qquad \text{"Quotient rule"} \quad (*) \qquad \text{if } g \neq 0 \,!$$

Proof of (*):

$$g \cdot \frac{f}{g} = f \qquad (\text{if } g \neq 0)$$

$$\Rightarrow \left(g \cdot \frac{f}{g}\right)' = f'$$

$$\Rightarrow g' \frac{f}{g} + g\left(\frac{f}{g}\right)' = f' \qquad \text{by product rule}$$

$$\Rightarrow \left(\frac{f}{g}\right)' = \frac{1}{g}\left(f' - g'\frac{f}{g}\right) = \frac{gf' - g'f}{g^2}$$

Chain rule:

$$\big(f(g(x))\big)' = f'(g(x))\, g'(x)$$

"outer derivative times inner derivative"

Proof:

$$\big(f(g(x))\big)' = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \qquad \underbrace{\frac{g(x+h) - g(x)}{h}} \xrightarrow[h \to 0]{} g'(x)$$

$$\frac{f(g(x)+k) - f(g(x))}{k} = \phi(k) \qquad k := g(x+h) - g(x)$$

If k is an arbitrary non-zero number, then

$$\lim_{k\to 0} \phi(k) = f'(g(x))$$

Moreover, by continuity of g, $\lim_{h\to 0} k(h) = 0$

so $h \to 0$ implies $k(h) \to 0$

This proof is almost complete, but we have to be careful that $k(h)$ might be 0.

Solution: extend $\phi(k)$ continuously by $f'(g(x))$ for $k=0$.

Examples: ① $f(x) = \cos(4x)$

$$f'(x) = \underset{\text{inner der.}}{4}\ \underset{\text{outer der.}}{(-\sin(4x))} = -4\sin(4x)$$

② $f(x) = x \Rightarrow f'(x) = 1$

③ $(xx)' = 1\cdot x + x \cdot 1 = 2x$

$(x\cdot x \cdot x)' = 1\cdot (x^2)' + x^2 \cdot 1 = 3x^2$

$\vdots$

$$\boxed{(x^n)' = n x^{n-1}} \qquad (*)$$

(*) Proof by induction:

$n=1$: $\quad x' = 1 = 1\cdot x^0$ ✓

$n \to n+1$: $\quad (x^{n+1})' = (x \cdot x^n)' = 1 \cdot x^n + x \underbrace{(x^n)'}_{n x^{n-1}} = (n+1)x^n$ ✓

④ $$f(x) = \frac{3x^2+1}{x^5+x}$$

$$f'(x) = \frac{3 \cdot 2x \cdot (x^5+x) - (5x^4+1)(3x^2+1)}{(x^5+x)^2} = \ldots\ldots$$

Q: can we extend the power rule (*) to rational exponents?

$$\left(x^{\frac{p}{q}}\right)' = \left(\left(u(x)\right)^p\right)' \qquad u(x) = x^{\frac{1}{q}}$$

$$= \underbrace{p\, u^{p-1}}_{\text{outer derivative}} \cdot \underbrace{u'(x)}_{\text{inner derivative}} \qquad (*) \qquad \text{chain rule}$$

$$\left(f(g(x))\right)' = f'(g(x))\, g'(x)$$

$u(x) = x^{\frac{1}{q}}$

$\Rightarrow u(x)^q = x$

$\Rightarrow q\, u(x)^{q-1}\, u'(x) = 1$ (**) chain rule on LHS

$$\frac{(*)}{(**)}: \quad \frac{\left(x^{\frac{p}{q}}\right)'}{1} = \frac{p\, u^{p-1}\, \cancel{u'}}{q\, u^{q-1}\, \cancel{u'}}$$

$$\Rightarrow \left(x^{\frac{p}{q}}\right)' = \frac{p}{q}\, u^{p-1-(q-1)}$$

$$= \frac{p}{q}\, u^{p-q}$$

$$= \frac{p}{q}\, x^{\frac{p}{q}-1}$$

$f(y) = y^p \qquad g(x) = u(x)$

$f(g(x)) = (u(x))^p$

$f'(y) = p\, y^{p-1} \qquad g'(x) = u'(x)$

$f'(g(x)) = p\,(u(x))^{p-1}$

$$\Rightarrow \boxed{\left(x^r\right)' = r\, x^{r-1}}$$

The power rule extends to rational numbers

This relation can be extended to arbitrary $r \in \mathbb{R}$ "by continuity".

(I.e. representing r as a limit of rational numbers.)