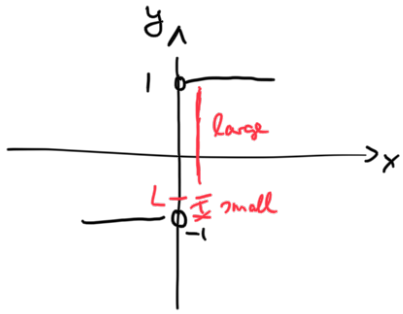
Recall from last lecture:

$\lim_{x \to x_0} f(x) = L$ if $f(x)$ can be made arbitrarily close to L by choosing x sufficiently close to x_0

Another example: $f(x) = \frac{|x|}{x}$



Domain $f = \mathbb{R} \setminus \{0\}$

$\lim_{x \to 0} f(x)$ does not exist: For every $x > 0$, $f(x) - f(-x) = 2$

so it's not possible to find a number L s.t.

$$|f(x) - L| < 1 = \varepsilon$$

no matter how close x is chosen to zero.

(In other words, for every $\delta > 0$, $|f(x) - L| + |f(-x) - L| \geq |f(x) - f(-x)| = 2$

for $|x - 0| < \delta$ in particular (but true for all x in fact).)

But it's possible to define "right limits" and "left limits":

$$\lim_{x \searrow x_0} f(x) = \lim_{\substack{x \to x_0 \\ x > x_0}} f(x) = \lim_{x \to x_0^+} f(x)$$

(Here: $\lim_{x \searrow 0} \frac{x}{|x|} = 1$)

$$\lim_{x \nearrow x_0} f(x) = \lim_{\substack{x \to x_0 \\ x < x_0}} f(x) = \lim_{x \to x_0^-} f(x)$$

(Here: $\lim \frac{x}{|x|} = -1$)

$x \to 0 \quad |x|$

So: $\lim_{x \to x_0} f(x)$ exist if and only if left and right limits exist and co-incide.

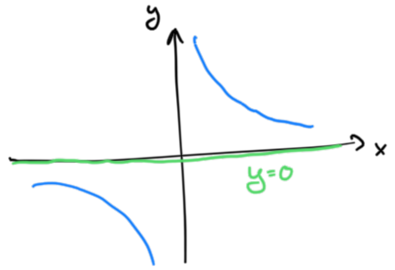
Horizontal asymptotes:

we write $\boxed{\lim_{x \to \infty} f(x) = L}$ if $\lim_{y \searrow 0} f\left(\frac{1}{y}\right) = L$

$\lim_{x \to -\infty} f(x) = L$ if $\lim_{y \nearrow 0} f\left(\frac{1}{y}\right) = L$

we say that the line $y = L$ is a horizontal asymptote

E.g.: ① $\lim_{x \to \infty} \frac{1}{x} = \lim_{y \searrow 0} \frac{1}{\frac{1}{y}} = \lim_{y \searrow 0} y = 0$



"as x gets large, f(x) gets arbitrarily close to 0"

like-wise $\lim_{x \to -\infty} \frac{1}{x} = 0$

$x = \frac{1}{y}$

$\frac{3}{\frac{1}{y}} = 3y$

② $\lim_{x \to \infty} \frac{x^2 + 3x - 2}{3x^2 - 2x + 5} = \lim_{x \to \infty} \frac{1 + \frac{3}{x} - \frac{2}{x^2}}{3 - \frac{2}{x} + \frac{5}{x^2}}$

$= \lim_{y \searrow 0} \frac{1 + 3y - 2y^2}{3 - 2y + 5y^2} = \frac{1}{3}$

Alternative:

$\lim_{x \to \infty} \frac{x^2 + 3x - 2}{3x^2 - 2x + 5} = \lim_{x \to \infty} \frac{x^2}{3x^2} = \lim_{x \to \infty} \frac{1}{3} = \frac{1}{3}$

Caution: You cannot "substitute" $x = \infty$ into an expression:

$$\lim_{x\to\infty} \frac{x}{x} = \lim_{x\to\infty} 1 = 1$$

$$\lim_{x\to\infty} \frac{x}{x^2} = \lim_{x\to\infty} \frac{1}{x} = 0 \quad \text{(by prev. example)}$$

$$\lim_{x\to\infty} \frac{x^2}{x} = \lim_{x\to\infty} x$$ is not defined because x is growing without bounds.

Vertical asymptotes:

If $f(x)$ increases (decreases) without bounds as $x \nearrow x_0$, we write

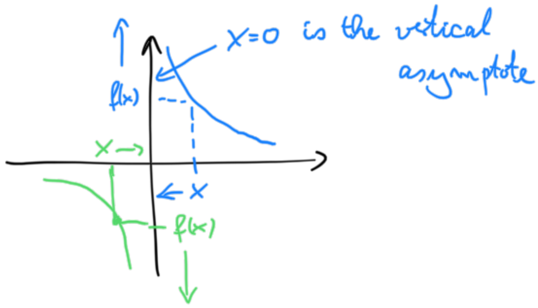
$$\lim_{x\nearrow x_0} f(x) = \infty \quad (-\infty)$$. We say that f has the vertical asymptote $x = x_0$

∞ is not a number, so we cannot compute (do algebraic manipulations) with this expression

Even if we write this, the limit does not exist !!

E.g.: ① $$\boxed{\lim_{x\searrow 0} \frac{1}{x} = \infty}$$

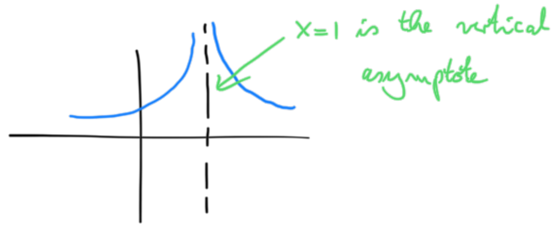
$$\boxed{\lim_{x\nearrow 0} \frac{1}{x} = -\infty}$$



② $$\lim_{x\searrow 1} \frac{1}{(x-1)^2} = \infty$$

$$\lim_{x\nearrow 1} \frac{1}{(x-1)^2} = \infty$$

Here: $$\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty$$



Continuity

Def: $f : (a,b) \to \mathbb{R}$

f is continuous at $x_0 \in (a,b)$ if $\boxed{\lim_{x \to x_0} f(x) = f(x_0)}$ $\quad (*)$

Possible ways that a function can be discontinuous:

(a) Non-removable discontinuities: vertical asymptotes, jumps, etc.

(b) Removable discontinuities:

$$\lim_{x \to x_0} f(x) = L \text{ exists, but } f(x_0) \neq L$$

Then, by <u>defining</u> $f(x_0) = L$, we can "remove" the discontinuity.

<u>Example for (b):</u>

$f(x) = \dfrac{x}{x}$, not defined at $x_0 = 0$,

but $\lim_{x \to 0} \dfrac{x}{x} = 1$, so <u>defining</u> $f(0) = 1$ makes f continuous.

so $\lim_{x \to 0} f(x) = 1 = f(0)$ This is what $(*)$ requires!

<u>Theorem:</u> $f : \mathbb{R} \to \mathbb{R}$ continuous

$g : (a,b) \to \mathbb{R}$, $\lim_{x \to x_0} g(x) = L$, $x_0 \in (a,b$

Then $\lim_{x \to x_0} f(g(x)) = f(L) = f\left(\lim_{x \to x_0} g(x)\right)$

"We can move limits inside of continuous functions"

<u>E.g.:</u>

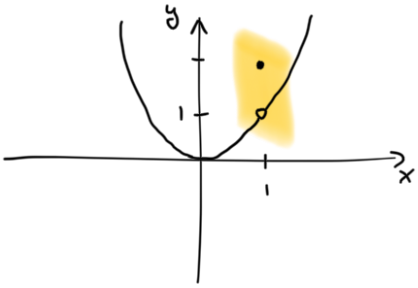
$$\lim_{x \to 3} \sin \frac{x^2 - 9}{x - 3} = \sin\left(\lim_{x \to 3} \underbrace{\frac{x^2-9}{x-3}}_{\frac{(x+3)(x-3)}{(x-3)} = x+3}\right) = \sin(6)$$

(Note: sin, cos are continuous functions, proof for some other time.)

Rule of thumb: "a function is continuous if you can draw it without lifting the pen"

E.g.:

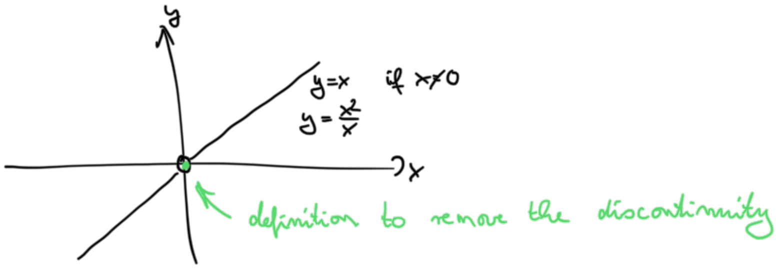
$$f(x) = \begin{cases} x^2 & \text{if } x \neq 1 \quad \leftarrow \text{continuous} \\ 2 & \text{if } x = 1 \end{cases}$$



discontinuity at $x = 1$, removable.

E.g.: $f(x) = \dfrac{x^2}{x}$

Remove the discontinuity by defining $f(0) := \lim_{x \to 0} \dfrac{x^2}{x} = \lim_{x \to 0} x = 0$



Note: There are functions that are discontinuous everywhere, e.g.

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{I} \end{cases}$$