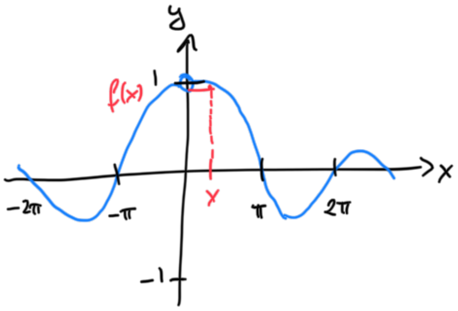
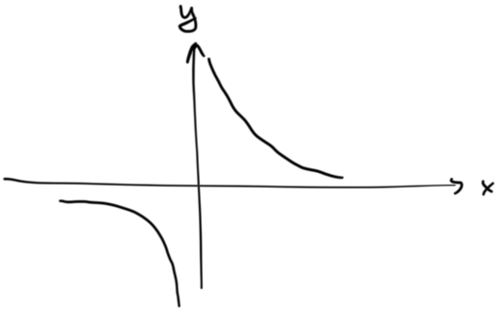
Limits of functions

Motivation: (a) $f(x) = \dfrac{\sin x}{x}$ domain: $\mathbb{R} \setminus \{0\}$



It appears that $f(x)$ gets arbitrarily close to $y=1$ as x gets close to zero.

(b) $f(x) = \dfrac{1}{x}$



It appears that $y=0$ is "a limiting value" of $f(x)$ as x gets large.

How to make this precise?

Def (informal): $L \in \mathbb{R}$ is the limit of $f(x)$ as x approaches x_0 if:

$f(x)$ can be made arbitrarily close to L for x sufficiently close to x_0.

We write: $\lim\limits_{x \to x_0} f(x) = L$ or $f(x) \xrightarrow[x \to x_0]{} L$

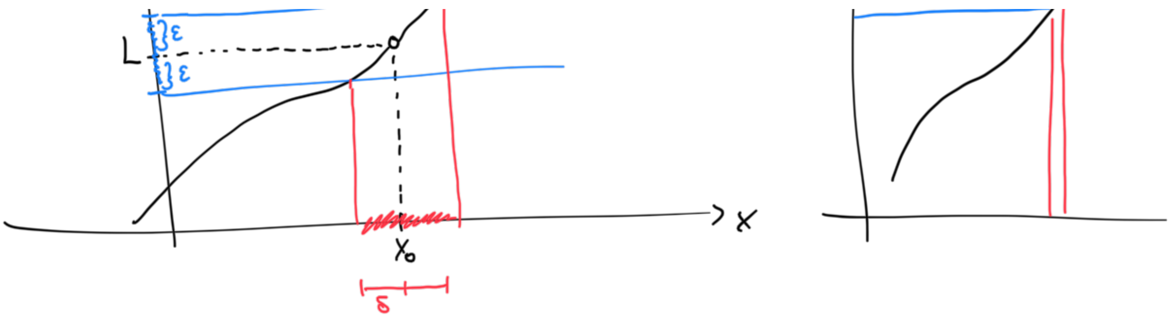
Def (formal): $x_0 \in (a,b)$, $f : (a,b) \setminus \{x_0\} \to \mathbb{R}$

Then $\lim\limits_{x \to x_0} f(x) = L$

if $\underset{\text{for all}}{\forall}\ \varepsilon > 0$ $\underset{\text{there exists}}{\exists}\ \delta > 0$ such that $\forall x \in (a,b) \setminus \{x_0\}$ with $0 < |x - x_0| < \delta$:

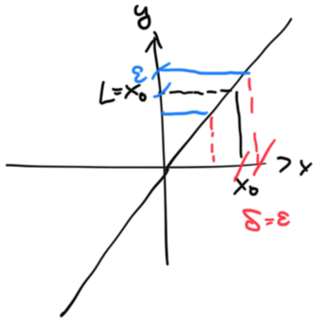
$|f(x) - L| < \varepsilon$

to ensure that $f(x)$ makes sense



δ is chosen as a function of ε

Examples: $f(x) = x$ guess: $L = x_0$



So given $\varepsilon > 0$, <u>choose</u> $\delta = \varepsilon$. (as a response to given ε)

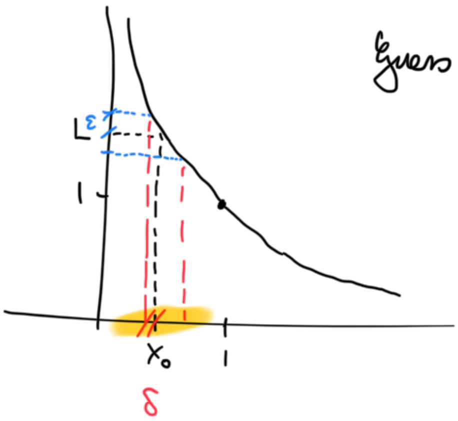
Then if $|x - x_0| < \delta = \varepsilon \Rightarrow |\underbrace{f(x)}_{=x} - \underbrace{L}_{x_0}| < \varepsilon$

So

$$\lim_{x \to x_0} x = x_0 .$$

② $f(x) = \frac{1}{x}$

Guess: $\lim_{x \to x_0} \frac{1}{x} = \frac{1}{x_0}$ $\qquad x_0 \neq 0$



For a given ε, how to choose δ?

Scratch calculation:

$\left| \frac{1}{x} - \frac{1}{x_0} \right| < \varepsilon$, need: $|x_0 - x| < |x x_0| \, \varepsilon$

$\underbrace{\frac{1}{x} - \frac{1}{x_0}}_{\frac{x_0 - x}{x_0 x}}$

To simplify: Assume $x, x_0 > 0$

Requirement on δ will be more stringent if $x < x_0$

So we need to "engineer" : $x_0 - x < \underbrace{x \, x_0}_{} \, \varepsilon$

$$= (x - x_0 + x_0) x_0 \varepsilon$$

$$= (x - x_0) x_0 \varepsilon + x_0^2 \varepsilon$$

$$\Leftrightarrow \underbrace{x_0 - x + (x_0 - x) x_0 \varepsilon}_{(x_0 - x)(1 + x_0 \varepsilon)} < x_0^2 \varepsilon$$

$$\Leftrightarrow x_0 - x < \boxed{\frac{x_0^2 \varepsilon}{1 + \varepsilon x_0}} = \delta$$

Now make a consistent argument:

Given $\varepsilon > 0$, $x_0 > 0$, choose $\delta = \frac{x_0^2 \varepsilon}{1 + \varepsilon x_0}$

Then for $|x_0 - x| < \delta$:

$$\Rightarrow |x_0 - x| < \frac{x_0^2 \varepsilon}{1 + \varepsilon x_0}$$

$$\Rightarrow |x_0 - x| \, (1 + \varepsilon x_0) < x_0^2 \varepsilon$$

$$\Rightarrow |x_0 - x| < \underbrace{x_0^2 \varepsilon - |x_0 - x| \, \varepsilon x_0}_{}$$

$$= x_0 \varepsilon \left(x_0 - |x_0 - x| \right)$$

$$\leq x_0 \varepsilon \, |x|$$

$a - |b| \leq |a - b|$

$\Leftrightarrow a \leq |a - b| + |b|$

$$\Rightarrow \frac{|x_0 - x|}{|x_0 x|} < \varepsilon$$

$$\Rightarrow \left| \frac{1}{x_0} - \frac{1}{x} \right| < \varepsilon$$

The limit laws:

Assume: $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$

Then: (i) $\lim_{x \to x_0} \left(f(x) + g(x) \right) = L + M$

(ii) $\lim_{x \to x_0} f(x)\, g(x) = LM$

(iii) $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$

$f(z) = u(z) + i v(z)$

Examples: ① $\lim_{x \to 0} \frac{x}{x} = \lim_{x \to 0} 1 = 1$

② $\lim_{t \to 0} \frac{\sqrt{t+25} - 5}{t} = \lim_{t \to 0} \frac{(\sqrt{t+25} - 5)(\sqrt{t+25} + 5)}{t\ (\sqrt{t+25} + 5)}$

$$= \lim_{t \to 0} \frac{\cancel{t} + \cancel{25} - \cancel{25}}{\cancel{t}\,(\sqrt{t+25} + 5)} = \frac{1}{\sqrt{0+25} + 5}$$

Note: we have used that $\lim_{x \to x_0} x^{\alpha} = x_0^{\alpha}$ $\alpha \geq 0$ (Here: $\alpha = \frac{1}{2}$)

③ $\lim_{s \to 2} \frac{s^2 - 4}{s^2 + s - 6} = \lim_{s \to 2} \frac{\cancel{(s-2)}(s+2)}{\cancel{(s-2)}(s+3)} = \lim_{s \to 2} \frac{s+2}{s+3} = \frac{2+2}{2+3} = \frac{4}{5}$