Equations and functions

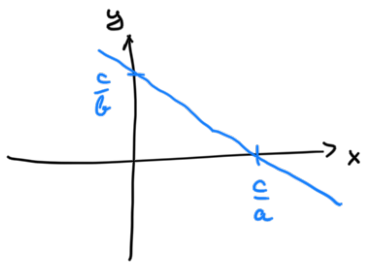
equations of two variables x and y are arbitrary relationships between x and y,

the graph of an equation is the set of all points (x,y) that satisfy the equation.

Examples: ① line

$$ax + by = c$$

a, b, c are given real numbers

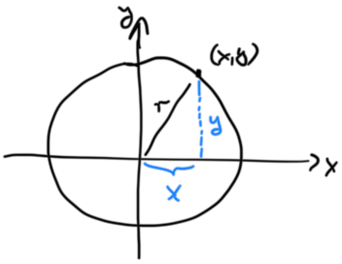


$a, b, c > 0$

In case $b \neq 0$, we can solve for y: $y = \frac{c}{b} - \frac{a}{b}x$ "y is a function of x"

" " $a \neq 0$, " " " " x: $x = \frac{c}{a} - \frac{b}{a}y$ "x is a function of y"

② circle: $x^2 + y^2 = r^2$



Q: can one of of the variables be expressed as a function of the other?

A: Not in this generality, but e.g. if we restrict to the upper half-circle $y \geq 0$:

$$y = \sqrt{r^2 - x^2}$$

Remark: But on upper semi-circle, cannot solve for x; to do this, restrict e.g. to right semi-circle.

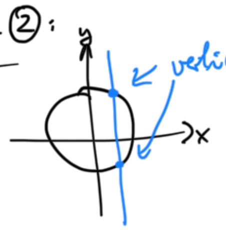
Def: A function $f: A \to B$ is a rule that assigns to any $x \in A$, A the domain, exactly one element $y \in B$.

- Range $f = \{f(x) : x \in A\} \subset B$ is a subset of B, but may be smaller

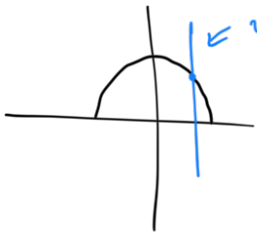
- The graph of f is the graph of the equation $y = f(x)$
- Every vertical line through a point $(x,0)$, $x \in A$, intersects the graph of f in exactly one point. "Vertical line test"

Back to example ②:

vertical line test fails



vertical line test succeeds



$$y = f(x) = \sqrt{r^2 - x^2}$$

Domain of f: $[-r, r]$ (if we want the range from within the real numbers.)

Range $f = [0, r]$

A zoo of functions:

① absolute value : For $z \in \mathbb{C}$, $|z| = \sqrt{z z^*}$

$$z = x + iy \qquad z^* = x - iy$$

$$\Rightarrow z z^* = x^2 + y^2$$

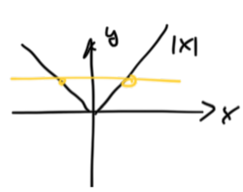
$$|z| = \sqrt{x^2 + y^2}$$

Range : $[0, \infty) \subset \mathbb{R}$

consistent

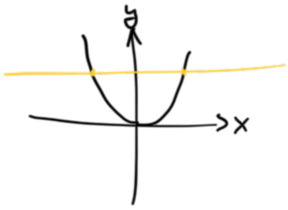
For $x \in \mathbb{R}$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



② Parabola

$y = x^2$



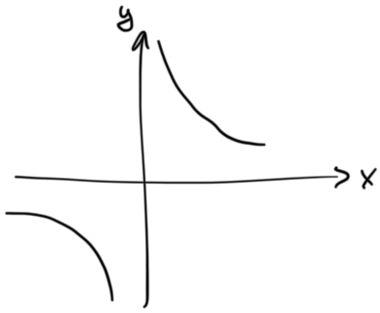
Function cannot be solved for x!

Note: y is a function of x, but not vice versa

Domain: $\mathbb{R}$ (or $\mathbb{C}$)

Range, if domain is $\mathbb{R}$: $[0,\infty) \subset \mathbb{R}$

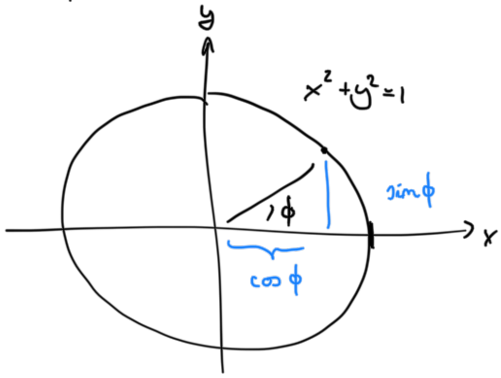
③ Hyperbola: $y = \frac{1}{x}$ $\left(x = \frac{1}{y} \right)$



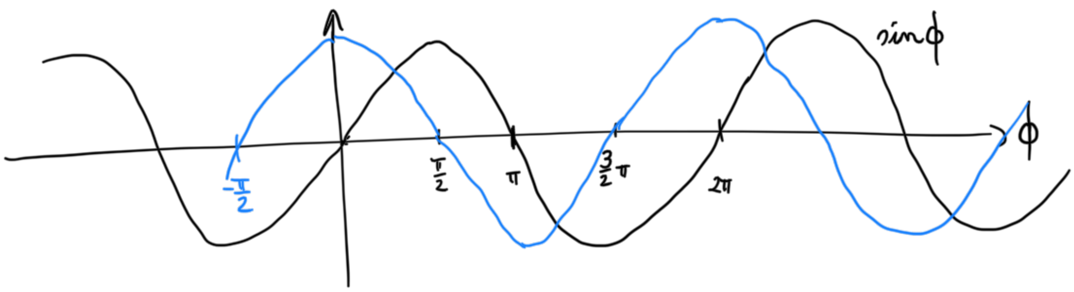
Domain: $\mathbb{R} \setminus \{0\}$

Range: $\mathbb{R} \setminus \{0\}$

④ Trigonometric functions:



$\sin^2\phi + \cos^2\phi = 1$



Domain of sin, cos is $\mathbb{R}$

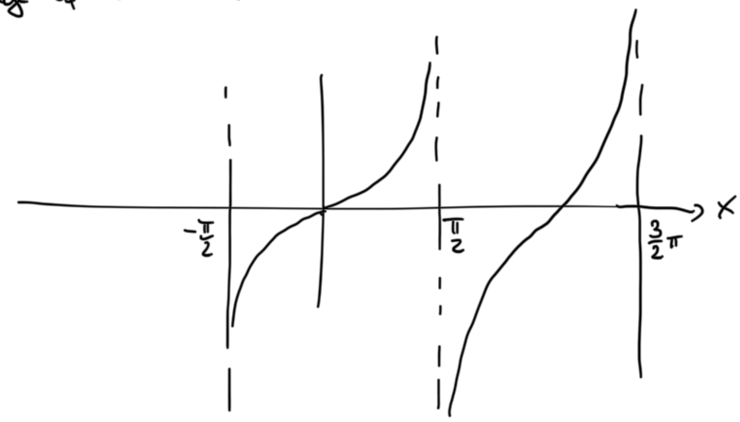
Range of sin, cos: $[-1,1]$

we can define: $\tan\phi = \dfrac{\sin\phi}{\cos\phi}$

Domain of tan: $\mathbb{R} \setminus \{ \frac{\pi}{2} + n\pi : n \in \mathbb{Z} \}$

"all real numbers except those where cos is zero"

Range of tan: $\mathbb{R}$



⑤ Exponential function:

$$a^n = \underbrace{a \cdot \cdots \cdot a}_{n \text{ times}} \qquad n \in \mathbb{N} \quad , \quad a \in \mathbb{R}, \ a > 0$$

$$a^{-n} = \frac{1}{a^n} \quad , \quad a^0 = 1 \qquad n \in \mathbb{N}$$

$$a^{\frac{p}{q}} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p \qquad q \in \mathbb{N}, \ p \in \mathbb{Z}$$

This definition can be extended to <u>real</u> exponents, and

$$\boxed{a^{x+y} = a^x a^y \qquad \text{for all } x,y \in \mathbb{R} \quad \text{and even } x,y \in \mathbb{C}}$$

$$\Rightarrow \quad (a^x)^y = a^{xy} = (a^y)^x \neq a^{(x^y)}$$

<u>Inverse of a function:</u>

If $f: A \to B$, then $g: B \to A$ is called the inverse of f if

$$g(f(x)) = x \qquad \text{for all } x \in A \qquad (*)$$

$$f(g(y)) = y \qquad \text{for all } y \in B \qquad (**)$$

In general: Range $f \subset B$

If Range $f = B$, then it suffices to check $(*)$, because if $(*)$ is true,

$$f(g(f(x))) = f(x)$$

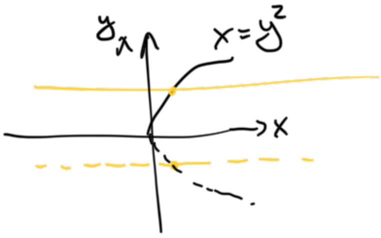
Know that for any $y \in B$, there is $x \in A$ s.t. $f(x) = y$.

$\Rightarrow$ If $y \in B$, take x s.t. $f(x) = y$

$\Rightarrow f(g(y)) = y$, so $(**)$ is a consequence of $(*)$.

The graph of f satisfies the horizontal line test if f is invertible (has an inverse):
it intersects any horizontal line through $(0, y)$, $y \in B$, exactly once.

E.g.:



satisfies horizontal line test

$$f(x) = \sqrt{x}$$

Inverse function: $g(y) = y^2$

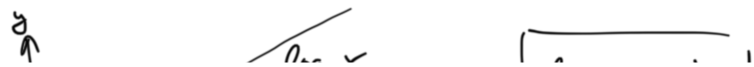
Inverse of exponential function:

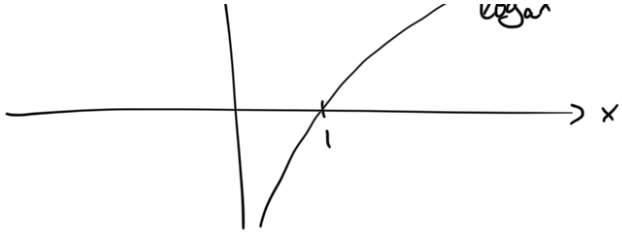
$$f(x) = a^x \qquad x > 0, \quad a > 0$$

then the inverse is called the logarithm to base a, we write

$$g(y) = \log_a y$$

$$\log_a a^x = x$$





$$\boxed{\log_a 1 = 0}$$

$$\log_a(xy) = \log_a x + \log_a y$$ (consequence of $a^{x+y} = a^x a^y$)

Example: $\sin x = y$