$\mathbb{C} \supset \mathbb{R}$ "$\mathbb{C}$ is a superset of $\mathbb{R}$"

or: every real number is a complex number

"Purely imaginary complex number": Real part is zero

Every polynomial can be factorized as

$$p(x) = a_n (x - z_1) \cdots (x - z_n)$$

where $z_1, \dots, z_n$ are n complex roots

$\deg(p) = n$

If one root of p is known, we can "divide out" this root and investigate the remaining polynomial of degree $n-1$.

Long division:

Example of integer division: $23 : 11 = 2{,}0909090\ldots = 2{,}\overline{09}$

$$\begin{aligned}
&23 : 11 = 2{,}0909090\ldots\\
-)\ &22\\
&\ \boxed{1}0\\
-)\ &\ \ 0\\
&\ \ \boxed{10}0\\
-)\ &\ \ \ 99\\
&\ \ \ \ \boxed{1}0\\
-)\ &\ \ \ \ \ 0\\
&\ \ \ \ \ 100
\end{aligned}$$

Remark: remainder $\square$ is always less than divisor, so only finitely many options

$\Rightarrow$ with integer division, this either terminates or becomes periodic

Division of polynomials:

Find all roots of $g(x) = 4x^3 + 3x^2 - 6x - 1$

By inspection, $z_1 = 1$ is a root, i.e. $x - 1$ is a factor of g

$$\begin{aligned}
&(4x^3 + 3x^2 - 6x - 1) : (x - 1) = 4x^2 + 7x + 1\\
-)\ &\ 4x^3 - 4x^2\\
&\qquad\ \ 7x^2 - 6x
\end{aligned}$$

$$
\begin{array}{r}
-\underline{|\ 7x^2 - 7x} \\
x - 1 \\
-\underline{|\ x - 1} \\
\underline{\underline{0}}
\end{array}
$$

zero means termination

Remaining roots, use quadratic formula

$$z_{2,3} = \frac{-7 \pm \sqrt{49 - 4 \cdot 4 \cdot 1}}{2 \cdot 4} = -\frac{7}{8} \pm \frac{\sqrt{33}}{8}$$

$\Rightarrow$ g has 3 real roots.

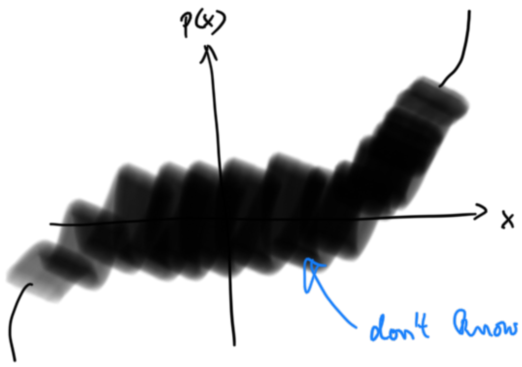
Note: A polynomial of odd degree with real coefficients has at least one real root.

Proof 1: If $z \notin \mathbb{R}$, then z^* is also a root, so such roots come in pairs.

Proof 2: For x very large, or $-x$ very large, the leading order term dominates. Since it is of odd degree, it's changing sign, so the polynomial has at least one real root.

Note: This needs the concept of limits, continuity, to be discussed.

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots \qquad a_n > 0$$



don't know precise behavior here

Inequalities: (by example)

$$-3 < \frac{1}{3}(7 - 2x) \leq 4$$

$$\Rightarrow \quad -9 < 7 - 2x \leq 12$$

$$\Rightarrow \quad -16 < -2x \leq 5$$

$[-\frac{5}{2}, 8[$ alternative notation

$\Rightarrow \quad 8 > x \geq -\frac{5}{2}$, in interval notation: $x \in [-\frac{5}{2}, 8)$

8 not included (parentheses)

$-\frac{5}{2}$ is included (bracket)

Example 2: $\underbrace{x^2 - 4x - 12}_{(x+2)(x-6)} > 0$ ($\geq$)

look for roots:

$$z_{1,2} = \frac{4 \pm \sqrt{16+48}}{2} = \frac{4 \pm 8}{2}$$

$z_1 = -2, \quad z_2 = 6$

This is satisfied if $x+2 > 0$, $x-6 > 0$ or $x+2 < 0$, $x-6 < 0$ ($\geq$ / $\leq$)

$\Rightarrow \quad x > 6$ or $x < -2$ ($\geq$ / $\leq$) $\quad x \in \mathbb{R} \setminus (-2, 6) = (-\infty, -2] \cup [6, \infty)$

in interval notation: $x \in \mathbb{R} \setminus [-2, 6] = (-\infty, -2) \cup (6, \infty)$

"without"

symbol for "no lower bound"

symbol for "no upper bound"

Message: Factorized forms of expressions contain more information