Recall: $\mathbb{N} = \{1, 2, 3, \dots\}$

$\mathbb{N}_0 = \{0, 1, 2, 3, \dots\} = \{0\} \cup \mathbb{N}$

$\mathbb{Z} = \{\dots -3, -2, -1, 0, 1, 2, \dots\} = -\mathbb{N} \cup \{0\} \cup \mathbb{N}$

$\mathbb{Q} = \{ \frac{p}{q} : p \in \mathbb{Z},\ q \in \mathbb{Z} \setminus \{0\} \}$

$\mathbb{I}$ = "numbers whose decimal representation is infinite and not eventually periodic"

$\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$ real numbers

Still: cannot solve all quadratic equations in the reals:

E.g.: $x^2 = -1$

Solution: define the imaginary unit i with the property that $i^2 = -1$

$$\mathbb{C} = \{ x + iy : x, y \in \mathbb{R} \}$$

Examples:

- $x^2 + 1 = \underbrace{(x+i)(x-i)}_{x^2 - (i)^2 = x^2 - (-1) = x^2 + 1}$

- $(7+3i)(1-i) = 7 - 7i + 3i + 3i(-i) = 7 + 3 + (-7+3)i = \underline{10 - 4i}$

- $\frac{3+i}{2-i} = \frac{3+i}{2-i} \cdot \frac{2+i}{2+i} = \frac{6+3i+2i-1}{2^2+1} = \frac{1}{5}(5+5i) = \underline{1+i}$

- Roots of $2x^2 - x + 3$:

$$x_{1,2} = \frac{1 \pm \sqrt{1^2 - 4 \cdot 2 \cdot 3}}{2 \cdot 2} = \frac{1 \pm \overbrace{\sqrt{-23}}^{= \sqrt{i^2 23} = i\sqrt{23}}}{4} = \frac{1}{4} \pm i \frac{\sqrt{23}}{4}$$

"complex conjugate pair"

The complex conjugate of $z = x + iy \in \mathbb{C}$ is $z^* = x - iy$ E.g. $(1+2i)^* = 1 - 2i$, $7^* = 7$

(x: real part, y: imaginary part)

sometimes it is written $\bar{z}$

Fact: If $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ has real coefficients, and z is a root, then z^* is also a root.

in general

Proof: $p(z^*) = a_n \underbrace{(z^*)^n}_{(z^n)^*} + a_{n-1} \underbrace{(z^*)^{n-1}}_{(z^{n-1})^*} + \ldots + a_0 \overset{\downarrow}{=} \left(a_n^* z^n + a_{n-1}^* z^{n-1} + \ldots + a_0^*\right)$

only equal if
$a_n = a_n^*$, so $a_n \in \mathbb{R}$
$a_{n-1} = a_{n-1}^*$, so $a_{n-1} \in \mathbb{R}$
$\vdots$

$= \left(a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0\right)^*$

$= p(z)^*$

$= 0^* = \underline{0}$

Why is $(z^n)^* = (z^*)^n$?

$z^2 = (x+iy)^2 = x^2 + 2ixy - y^2$

$z^3, \ldots, z^n$

Example: $i^* = -i$

$(i^2)^* = (-1)^* = -1$, $(i^*)^2 = (-i)^2 = i^2 = -1$

$(i^3)^* = (-i)^* = i$ $\qquad (i^*)^3 = (-i)^3 = -1 \cdot \underbrace{i \cdot i}_{=-1} \cdot i = i$

$\vdots$

coffee break exercise: finish the argument

More on roots of polynomials:

$n=1$: always one real root

$n=2$: $a_2 x^2 + a_1 x^1 + a_0 = 0$ $\qquad \Delta = a_1^2 - 4a_2 a_0 \begin{cases} >0 & \text{two distinct real roots} \\ =0 & \text{one "double" root} \\ <0 & \text{complex conjugate pair of roots} \end{cases}$

$n=3,4$: formulas available, but rarely useful (too messy)

$n \geq 5$: no general formula exists

Fundamental theorem of algebra:

Every polynomial of degree ≥ 1 has at least one complex root.

Consequence: Any polynomial can be written in factorized form as

$$p(x) = a_n (x - z_1)(x - z_2)(x - z_3) \cdot \ldots \cdot (x - z_n)$$

where $z_1 \ldots z_n$ are n complex roots of p (repetitions allowed)

... times, we speak of a root of order k, e.g. "double root", "triple root"...)

(If one of these appears k-times, we spea…

(complex) number

Reason: $p(x) = (x-\alpha)\,q(x) + r$ This is always possible (long division)

polynomial of degree $n-1$

If α is a root of p, then $p(\alpha) = 0 \Rightarrow \underbrace{(\alpha-\alpha)\,q(\alpha)}_{=0} + r = 0 \Rightarrow r = 0$

so $p(x) = (x-\alpha)\,q(x)$

Remark: $\mathbb{Q}$ is countable

$\mathbb{I}$ is uncountable

(Discussion in Analysis I)