Calculus and Elements of Linear Algebra I

Homework 9

Due on Moodle, Monday, November 16, 2020

1. Radiocarbon dating is a method to determine the age of objects composed of organic material. It is based on the fact that atmospheric carbon is composed of a fixed ratio of two stable carbon isotopes, ^{12}C and ^{13}C, and an unstable isotope, ^{14}C, which undergoes radioactive decay with a half-life of 5 730 years. (This ratio is maintained by the interaction of cosmic rays with atmospheric nitrogen.) Living organisms exchange carbon with the atmosphere sufficiently fast so that the carbon ratio in their tissue is close to the atmospheric ratio. When an organism dies, the amount of stable carbon in its organic matter remains constant while ^{14}C decays, so that the age can be estimated by measuring the "carbon ratio", the ratio of the number of ^{14}C to the number of stable carbon atoms.

   An archaeological object is found to have a carbon ratio which is 74% of the atmospheric carbon ratio. Determine the age of the object.

2. Frictional forces are typically proportional to the velocity, i.e.,
$$F(v) = -k\,v$$
   where k is a positive constant. Now assume that a particle of mass m subject to a frictional force has an initial velocity v_0 and initial position $x_0 = 0$.

   (a) How does the velocity of the particle change with time? Does it come to a standstill in a finite time?

   Hint: Use Newton's second law $F = m\,a = m\,\mathrm{d}v/\mathrm{d}t$.

   (b) Does the particle travel a finite or an infinite distance?

   (c) Use the definition of work,
$$W = \int_{x_0}^{x_{\text{final}}} F(x)\,\mathrm{d}x\,,$$
   to compute the work done by frictional forces on the particle. Interpret the result.

3. Solve the differential equation
$$\frac{\mathrm{d}y}{\mathrm{d}x} = y\,(2x+1)\,,$$
$$y(0) = 2\,.$$

4. Solve the differential equation

$$
\begin{aligned}
\mathrm{e}^x\,\mathrm{e}^y\,\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{e}^x\,\mathrm{e}^y &= x^2\,,\\
y(0) &= 0\,.
\end{aligned}
$$

Hint: Recognize the left hand side of the equation as the derivative of some expression, then integrate.

5. For each of the following equations, determine the equilibrium points where $y'(x) = 0$ and classify each as stable (y' changes sign from positive to negative at x) or unstable (y' changes sign from negative to positive at x). Sketch a few solution curves (without trying to solve the equation) in the x-y plane.

   (a) $y' = y - y^2$

   (b) $y' = y\,(y-1)\,(y-2)$

   (c) $y' = \mathrm{e}^y - 1$