Calculus and Elements of Linear Algebra I

Homework 10

Due on Moodle, Monday, November 23, 2020

1. Prove the following identities for vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3$.

   (a) The *"BAC–CAB-identity"*

   $$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}\,(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}\,(\boldsymbol{a} \cdot \boldsymbol{b})\,.$$

   (b) The *Jacobi identity* in three dimensions

   $$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = 0\,.$$

2. Prove the following identities for vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \mathbb{R}^3$.

   (a) The *Cauchy–Binet formula* in three dimensions

   $$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})\,(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})\,(\boldsymbol{b} \cdot \boldsymbol{c})\,.$$

   Hint: Use the identity $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \boldsymbol{v} \cdot (\boldsymbol{w} \times \boldsymbol{u})$.

   (b) The identity

   $$\|\boldsymbol{a} \times \boldsymbol{b}\|^2 = \|\boldsymbol{a}\|^2\,\|\boldsymbol{b}\|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2\,.$$

3. Find an equation for the plane that contains the point $\boldsymbol{p} = (2, 4, 6)$ and the line

   $$\boldsymbol{x} = \begin{pmatrix} 7 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}\,.$$

4. Find the distance between the point $\boldsymbol{p} = (1, 2, 3)$ and the line

   $$\boldsymbol{x} = \begin{pmatrix} -1 \\ 1 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\,.$$

5. Prove the following statement: Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ be linearly independent. If a vector $\boldsymbol{w}$ can be written

   $$\boldsymbol{w} = \sum_{k=1}^{n} \alpha_k\,\boldsymbol{v}_k\,,$$

   the choice of the coefficients $\alpha_1, \ldots, \alpha_n$ is unique.