Nonlinear Programming

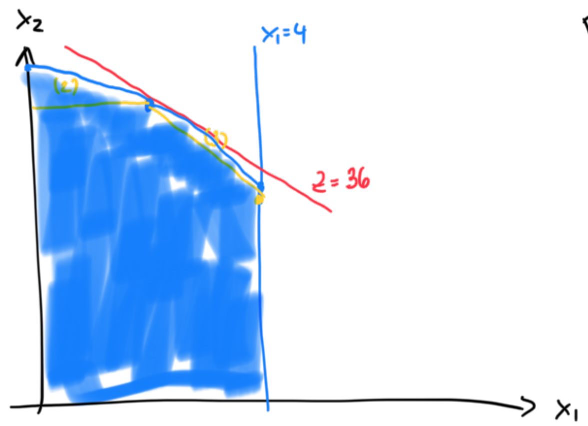
Recall: WYNDOOR example:

$$\text{maximize} \quad z = 3x_1 + 5x_2$$

$$\begin{aligned} \text{subject to} \quad x_1 \quad &\le 4 \quad &(1)\\ 2x_2 &\le 12 \quad &(2)\\ 3x_1 + 2x_2 &\le 18 \quad &(3)\\ x_1, x_2 &\ge 0 \end{aligned}$$

① Nonlinear constraint: replace (2), (3) by

$$9x_1^2 + 5x_2^2 \le 216$$

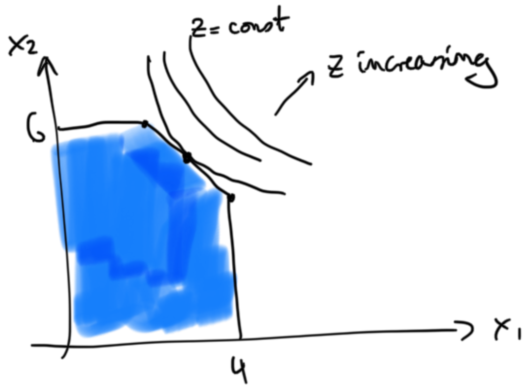


! Optimal value is not taken at a corner point of the feasible region.

② Nonlinear objective function, linear constraints:

$$\text{maximize} \quad z = 126x_1 - 9x_1^2 + 182x_2 - 13x_2^2$$

subject to (1), (2), (3)



! optimal value may be taken at an interior point of an edge, even if constraints are linear.

③ Maximize $z = 54x_1 - 9x_1^2 + 78x_2 - 13x_2^2$

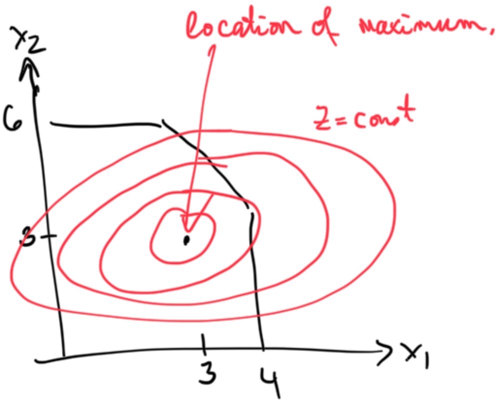
subject to (1), (2), (3).

Without constraints:

$$\frac{\partial z}{\partial x_1} = 54 - 18x_1 = 0 \quad \Rightarrow \quad x_1 = \frac{54}{18} = 3$$

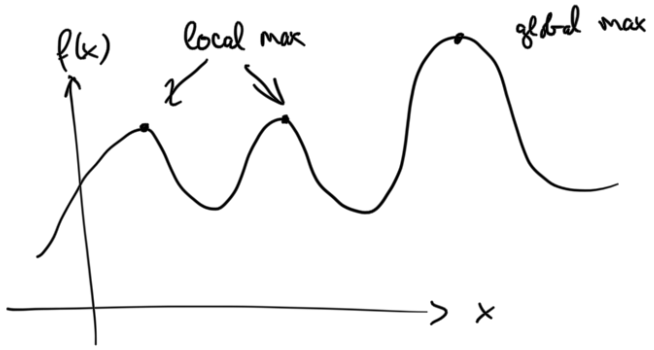
$$\frac{\partial z}{\partial x_2} = 78 - 26x_2 = 0 \quad \Rightarrow \quad x_2 = \frac{78}{26} = 3$$

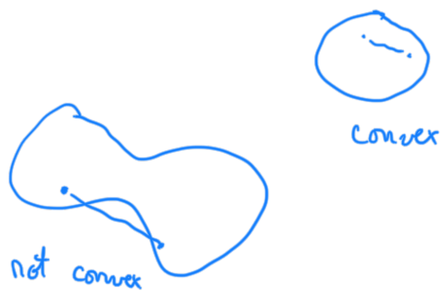
Note: $z(x_1, x_2)$ is composed of parabolas open downward, so expect to find at least one maximum. So the only candidate point $(x_1, x_2) = (3,3)$ corresponds to a maximum.



! Optimal solution may even occur in the interior of the feasible region.

Another problem: Local maximum may not be a global maximum.





When is a local maximum a global maximum?

One sufficient criterion:

(i) the feasible region is convex
(take any two feasible points, line segment between them stays within the feasible region)

(ii) the objective function is convex (for a minimum)
or concave (for a maximum)

→ "convex optimization", solvers [illegible]

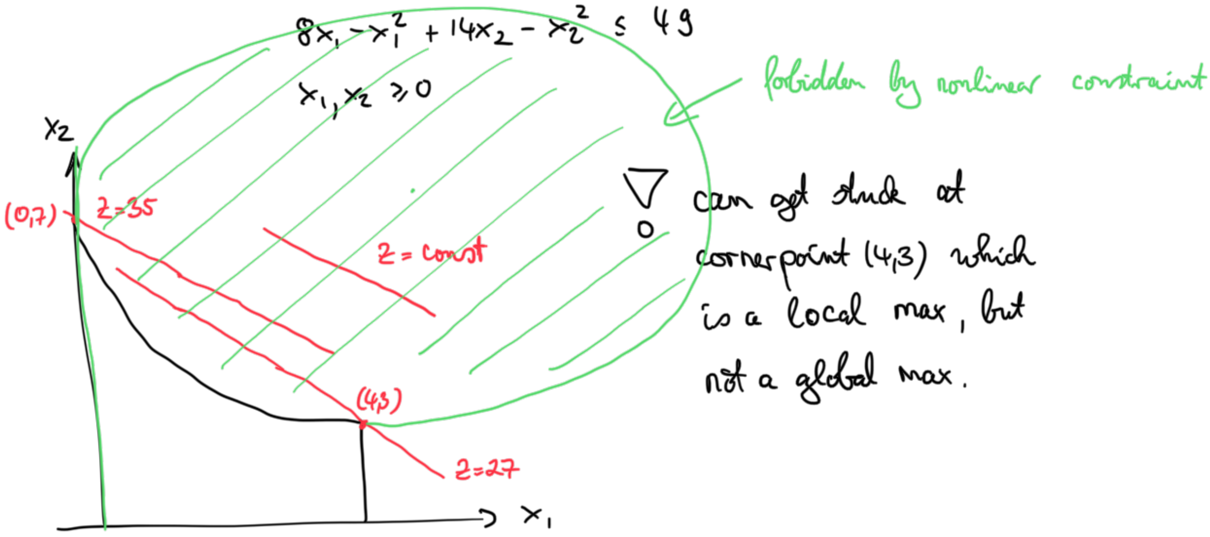
(4) Convexity is violated:

maximize $\quad z = 3x_1 + 5x_2$

subject to $\quad x_1 \le 4 \qquad\qquad x_2 \le 7$

$$8x_1 - x_1^2 + 14x_2 - x_2^2 \le 49$$

$$x_1, x_2 \ge 0$$



⚠ can get stuck at cornerpoint (4,3) which is a local max, but not a global max.

Types of nonlinear programs:

1. unconstrained optimization

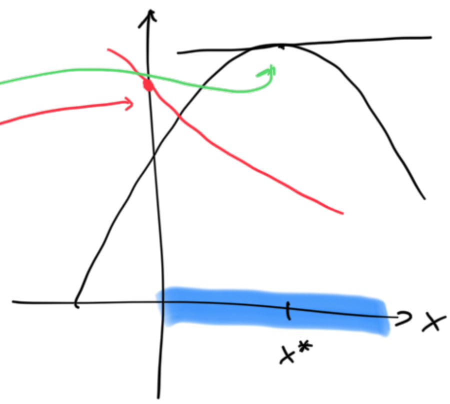
   max $f(x) \qquad\qquad x \in \mathbb{R}^n$

   necessary condition for having an optimum at $x = x^*$, if f is differentiable

   $$\frac{\partial f}{\partial x_j} = 0 \quad \text{at } x = x^*$$

   with trivial non-negativity constraints: $\quad x \ge 0$

   $$\frac{\partial f}{\partial x_j} \begin{cases} = 0 & \text{if } x_j^* > 0 \\ \le 0 & \text{if } x_j^* = 0 \end{cases}$$



2. for general nonlinear programs with differentiable objective function and constraint functions, neccesary conditions exist (KKT conditions) "Karush - Kuhn - Tucker"

3. Quadratic programming: f quadratic, constraints linear

KKT conditions $\to$ linear complementarity problem $\to$ modified simplex

4. Convex programming:

$$\max f(x)$$
$$\text{subject to } g_i(x) \leq b_i$$
$$x \geq 0$$

f concave

g_i convex

then local max is a global max. (good for solvers like ipopt)

5. Fractional programming: (here "fractional linear programming")

$$\max f(x) = \frac{c^T x + c_0}{d^T x + d_0}$$

$c \geq 0$, $d \geq 0$, $d_0 > 0$

$$\text{subject to } Ax \leq b$$
$$x \geq 0$$

Trick: $y = \dfrac{x}{d^T x + d_0}$ $\quad t = \dfrac{1}{d^T x + d_0}$

then: $\max \; c^T y + c_0 t$

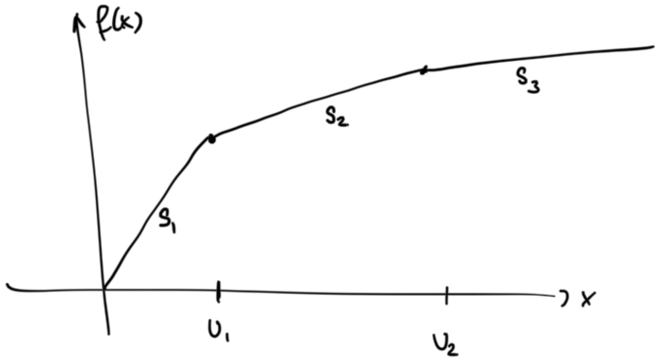
$$\text{subject to } Ay \leq tb$$
$$d^T y + d_0 t = 1$$

$y \geq 0$
$t \geq 0$

6. Linearly separable programs:

$$f(x) = \sum_{j=1}^{n} f_j(x_j) \qquad x \in \mathbb{R}^n$$

Often: $f_j(x_j)$ is piece-wise linear



Idea: $x_j = x_{j1} + x_{j2} + x_{j3}$

$0 \le x_{j1} \le U_{j1}$, $0 \le x_{j2} \le U_{j2}$, ...

In general: $x_{j(i+1)} = 0$ if $x_{ji} < U_{ji}$

$f_j(x_j) = s_{j1} x_{j1} + s_{j2} x_{j2} + ...$