# Dynamic Programming

- Problem can be divided into "stages"
- decision variable at each stage $x_i$, is a policy decision to transition from state $s$ at stage $i-1$ to state $x_i$ at the end of stage $i$

Example: STAGECOACH Problem

Let $f_i(s, x_i)$ denote the cost of travel (insurance) for going from $s$ at stage $i-1$ all the way to the end, transitioning to $x_i$ at stage $i$.

$f_i^*(s)$ denote the optimal cost starting from $s$ at stage $i-1$

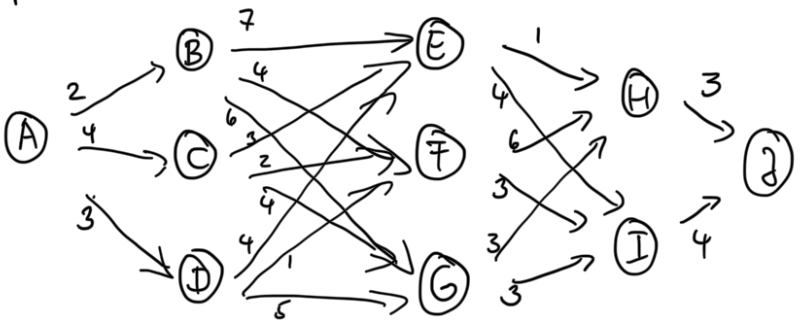
$x_i^*$ denote the optimal choice of next state at stage $i$.

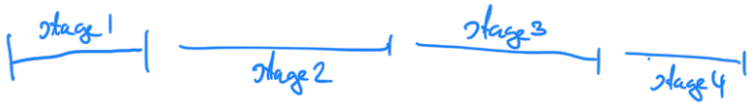
$$f_i(s, x_i) = \underbrace{c_{s,x_i}}_{\text{cost of travel at stage } i \text{ from } s \text{ to } x_i} + \underbrace{f_{i+1}^*(x_i)}_{\text{optimal future cost}}$$

$$f_i^*(s) = \min_{x_i} f_i(s, x_i)$$

If this structure is present, can solve the problem stage by stage in reverse order.

For stagecoach problem




$n=4$:

| $s$ | $f_4^*$ | $x_4^*$ |
|---|---|---|
| I | 4 | J |
| H | 3 | J |

$n=3$:

| S | $f_3(s,x_3) = c_{s,x_3} + f_4^*(x_3)$ $x_3=I$ | $x_3=H$ | $f_3^*(s)$ | $x_3^*$ |
|---|---|---|---|---|
| E | 4+4=8 | 1+3=4 | 4 | H |
| F | 3+4=7 | 6+3=9 | 7 | I |
| G | 3+4=7 | 3+3=6 | 6 | H |

$n=2$:

| S | $f_2(s,x_2) = c_{s,x_2} + f_3^*(x_2)$ $x_2=E$ | $x_2=F$ | $x_2=G$ | $f_2^*(s)$ | $x_2^*$ |
|---|---|---|---|---|---|
| B | 7+4=11 | 4+7=11 | 6+6=12 | 11 | E or F |
| C | 3+4=7 | 2+7=9 | 4+6=10 | 7 | E |
| D | 4+4=8 | 1+7=8 | 5+6=11 | 8 | E or F |

$n=1$:

| S | $f_1(s,x_1) = c_{s,x_1} + f_2^*(x_1)$ $x_1=B$ | $x_1=C$ | $x_1=D$ | $f_1^*$ | $x_1^*$ |
|---|---|---|---|---|---|
| A | 2+11=13 | 4+7=11 | 3+8=11 | 11 | C or D |

$\Rightarrow$ minimal cost is 11, on the paths

A-C-E-H-} or A-D-E-H-} or A-D-F-I-}

Example2: "World Health Council Problem"

Thousands of additional person-years of life

| Medical Teams | Countries 1 | 2 | 3 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 45 | 20 | 50 |
| 2 | 70 | 45 | 70 |
| 3 | 90 | 75 | 80 |
| 4 | 105 | 110 | 100 |
| 5 | 120 | 150 | 130 |

Q: distribute the 5 teams to the 3 countries to maximize person-years saved.

$x_i$: # of teams sent to country $i$

s : # of teams still available

$n = 3$:

| s | $f_3^*$ | $x_3^*$ |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 50 | 1 |
| 2 | 70 | 2 |
| 3 | 80 | 3 |
| 4 | 100 | 4 |
| 5 | 130 | 5 |

$n = 2$:

$f_2(s, x_2) = b_{s,x_2} + f_3^*(s - x_2)$

| s | $x_2=0$ | $x_2=1$ | $x_2=2$ | $x_2=3$ | $x_2=4$ | $x_2=5$ | $f_2^*(s)$ | $x_2^*$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0+0 | — | — | — | — | — | 0 | 0 |
| 1 | 0+50 | 20+0 | — | — | — | — | 50 | 0 |
| 2 | 0+70 | 20+50 | 45+0 | — | — | — | 70 | 0 or 1 |
| 3 | 0+80 | 20+70 | 45+50 | 75+0 | — | — | 95 | 2 |
| 4 | 0+100 | 20+80 | 45+70 | 75+50 | 110+0 | — | 125 | 3 |
| 5 | 0+130 | 20+100 | 45+80 | 75+70 | 110+50 | 150+0 | 160 | 4 |

$n = 1$:

$f_1(5, x_1) = b_{5,x_1} + f_2^*(5 - x_1)$

| s | $x_1=0$ | $x_1=1$ | $x_1=2$ | $x_1=3$ | $x_1=4$ | $x_1=5$ | $f_1^*$ | $x_1^*$ |
|---|---|---|---|---|---|---|---|---|
| 5 | 0+160 | 45+125 = 170 | 70+95 = 165 | 90+70 = 160 | 105+50 = 155 | 120+0 = 120 | 170 | 1 |

Solution: Send 1 team to country 1, 3 teams to country 2, 1 team to country 3 for a total benefit of 170 000 person-years of life saved.