Motivating example for the transportation problem

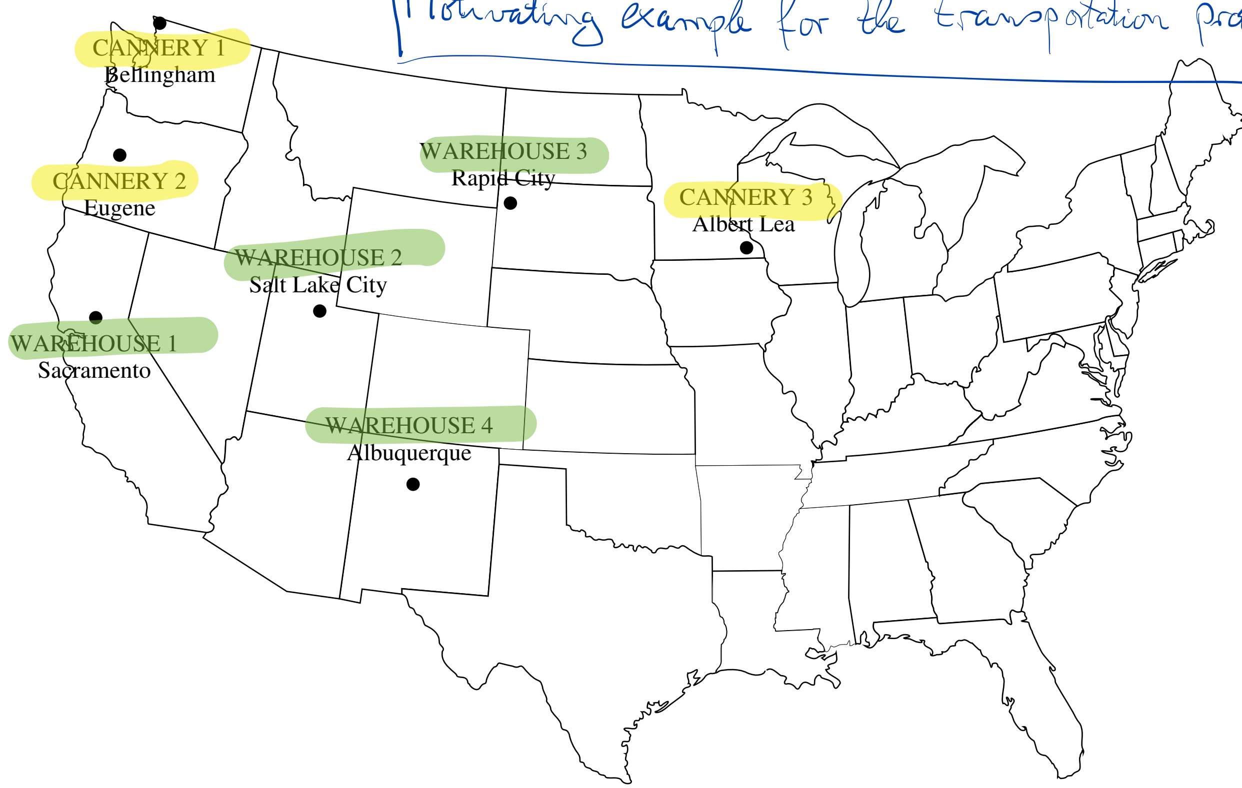

FIGURE 8.1
Location of canneries and warehouses for the P & T Co. problem.

Data required by the transportation problem

TABLE 8.2 Shipping data for P & T Co.

	Shipping Cost ($) per Truckload				
	Warehouse				
	1	2	3	4	S_i Output
Cannery 1	464 c_{11}	513 c_{12}	654	867	75 s_1
Cannery 2	352	416	690	791	125 s_2
Cannery 3	995	682	388	685	100 s_3
Allocation d_j	80 d_1	65 d_2	70 d_3	85 d_4	

decision variables: x_{ij} i: cannery j: warehouse cost data: c_{ij} unit cost of transport from cannery i to warehouse j

Network view of the transportation problem

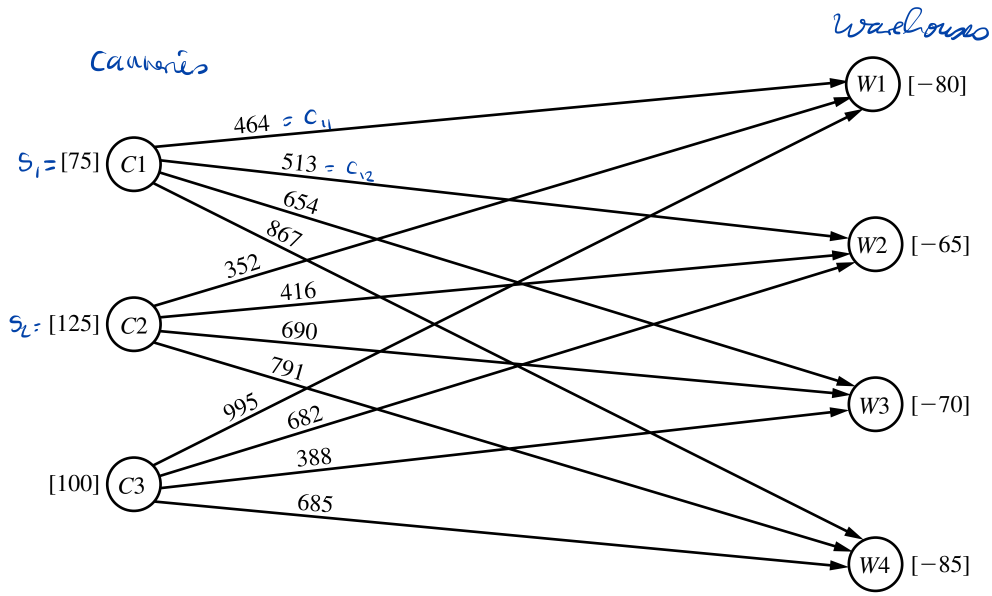

FIGURE 8.2
Network representation of the P & T Co. problem.

subject to the constraints

$$
\begin{array}{l}
x_{11}+x_{12}+x_{13}+x_{14} = 75\\
x_{21}+x_{22}+x_{23}+x_{24} = 125\\
x_{31}+x_{32}+x_{33}+x_{34} = 100\\
x_{11}+x_{21}+x_{31} = 80\\
x_{12}+x_{22}+x_{32} = 65\\
x_{13}+x_{23}+x_{33} = 70\\
x_{14}+x_{24}+x_{34} = 85
\end{array}
$$

and

$$x_{ij} \geq 0 \qquad (i = 1, 2, 3;\ j = 1, 2, 3, 4).$$

Table 8.3 shows the constraint coefficients. As you will see later in this section, it is the special structure in the pattern of these coefficients that distinguishes this problem as a transportation problem, not its context.

TABLE 8.3 Constraint coefficients for P & T Co.

	Coefficient of:												
	x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	x_{31}	x_{32}	x_{33}	x_{34}	
A =	1	1	1	1									Cannery constraints
					1	1	1	1					
									1	1	1	1	
	1				1				1				Warehouse constraints
		1				1				1			
			1				1				1		
				1				1				1	

TABLE 8.5 Parameter table for the transportation problem

		Cost per Unit Distributed				
		Destination				
		1	**2**	⋯	***n***	**Supply**
Source	1	c_{11}	c_{12}	⋯	c_{1n}	s_1
	2	c_{21}	c_{22}	⋯	c_{2n}	s_2
	⋮	⋯	⋯	⋯	⋯	⋮
	m	c_{m1}	c_{m2}	⋯	c_{mn}	s_m
Demand		d_1	d_2	⋯	d_n	

Therefore, formulating a problem as a transportation problem only requires filling out a parameter table in the format of Table 8.5. Alternatively, the same information can be provided by using the network representation of the problem shown in Fig. 8.3. It is not necessary to write out a formal mathematical model.

However, we will go ahead and show you this model once for the general transportation problem just to emphasize that it is indeed a special type of linear programming problem.

Letting Z be the total distribution cost and x_{ij} ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$) be the number of units to be distributed from source i to destination j, the linear programming formulation of this problem is

Mathematical formulation of the transportation problem

$$\text{Minimize} \qquad Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij},$$

cost of transporting x_{ij} units at c_{ij} cost/unit from i to j

subject to

$$\sum_{j=1}^{n} x_{ij} = s_i \qquad \text{for } i = 1, 2, \ldots, m,$$

$$\sum_{i=1}^{m} x_{ij} = d_j \qquad \text{for } j = 1, 2, \ldots, n,$$

i : cannery

j : warehouse

and

$$x_{ij} \geq 0, \qquad \text{for all } i \text{ and } j.$$

Note that the resulting table of constraint coefficients has the special structure shown in Table 8.6. *Any* linear programming problem that fits this special formulation is of the transportation problem type, regardless of its physical context. In fact, there have been numerous applications unrelated to transportation that have been fitted to this special structure, as we shall illustrate in the next example later in this section. (The assignment problem described in Sec. 8.3 is an additional example.) This is one of the reasons why the transportation problem is considered such an important special type of linear programming problem.

Second example:

M: very large number

TABLE 8.10 Water resources data for Metro Water District

	Cost (Tens of Dollars) per Acre Foot				Total supply: 160
	Berdoo	**Los Devils**	**San Go**	**Hollyglass**	**Supply**
Colombo River	16	13	22	17	50
Sacron River	14	13	19	15	60
Calorie River	19	20	23	— M	50
Total: 110 Minimum needed	30	70	0	10	(in units of 1 million acre feet)
Total: 210 Requested	50	70	30	~~∞~~ 60	

Total requested > Total supply => Add dummy source 50

the water and the city being supplied. The variable cost per acre foot of water (in tens of dollars) for each combination of river and city is given in Table 8.10. Despite these variations, the price per acre foot charged by the district is independent of the source of the water and is the same for all cities.

The management of the district is now faced with the problem of how to allocate the available water during the upcoming summer season. In units of 1 million acre feet, the amounts available from the three rivers are given in the rightmost column of Table 8.10. The district is committed to providing a certain minimum amount to meet the essential needs of each city (with the exception of San Go, which has an independent source of water), as shown in the *minimum needed* row of the table. The *requested* row indicates that Los Devils desires no more than the minimum amount, but that Berdoo would like to buy as much as 20 more, San Go would buy up to 30 more, and Hollyglass will take as much as it can get.

Management wishes to allocate *all* the available water from the three rivers to the four cities in such a way as to at least meet the essential needs of each city while minimizing the total cost to the district.

Formulation. Table 8.10 already is close to the proper form for a parameter table, with the rivers being the sources and the cities being the destinations. However, the one basic difficulty is that it is not clear what the demands at the destinations should be. The amount to be received at each destination (except Los Devils) actually is a decision variable, with both a lower bound and an upper bound. This upper bound is the amount requested unless the request exceeds the total supply remaining after the minimum needs of the other cities are met, in which case this *remaining supply* becomes the upper bound. Thus, insatiably thirsty Hollyglass has an upper bound of

$$(50 + 60 + 50) - (30 + 70 + 0) = 60.$$

Unfortunately, just like the other numbers in the parameter table of a transportation problem, the demand quantities must be *constants,* not bounded decision variables. To begin resolving this difficulty, temporarily suppose that it is not necessary to satisfy the minimum needs, so that the upper bounds are the only constraints on amounts to be allocated to the cities. In this circumstance, can the requested allocations be viewed as the demand quantities for a transportation problem formulation? After one adjustment, yes! (Do you see already what the needed adjustment is?)

Revised parameter table for the second example:

TABLE 8.12 Parameter table for Metro Water District

			Cost (Tens of Millions of Dollars) per Unit Distributed					
			Destination					
			Berdoo (min.) 1	Berdoo (extra) 2	Los Devils 3	San Go 4	Hollyglass 5	Supply
Source	Colombo River	1	16	16	13	22	17	50
	Sacron River	2	14	14	13	19	15	60
	Calorie River	3	19	19	20	23	M	50
	Dummy	4(D)	M	0	M	0	0	50
Demand			30	20	70	30	60	

This problem will be solved in the next section to illustrate the solution procedure presented there.

8.2 A STREAMLINED SIMPLEX METHOD FOR THE TRANSPORTATION PROBLEM

Because the transportation problem is just a special type of linear programming problem, it can be solved by applying the simplex method as described in Chap. 4. However, you will see in this section that some tremendous computational shortcuts can be taken in this method by exploiting the special structure shown in Table 8.6. We shall refer to this streamlined procedure as the **transportation simplex method.**

As you read on, note particularly how the special structure is exploited to achieve great computational savings. This will illustrate an important OR technique—streamlining an algorithm to exploit the special structure in the problem at hand.

Setting Up the Transportation Simplex Method

To highlight the streamlining achieved by the transportation simplex method, let us first review how the general (unstreamlined) simplex method would set up a transportation problem in tabular form. After constructing the table of constraint coefficients (see Table 8.6), converting the objective function to maximization form, and using the Big M method to introduce artificial variables $z_1, z_2, \ldots, z_{m+n}$ into the $m + n$ respective equality constraints (see Sec. 4.6), typical columns of the simplex tableau would have the form shown in Table 8.13, where all entries *not shown* in these columns are *zeros.* [The one remaining adjustment to be made before the first iteration of the simplex method is to algebraically eliminate the nonzero coefficients of the initial (artificial) basic variables in row 0.]

After any subsequent iteration, row 0 then would have the form shown in Table 8.14. Because of the pattern of 0s and 1s for the coefficients in Table 8.13, by the *fundamental insight* presented in Sec. 5.3, u_i and v_j would have the following interpretation:

u_i = multiple of *original* row i that has been subtracted (directly or indirectly) from *original* row 0 by the simplex method during all iterations leading to the current simplex tableau.