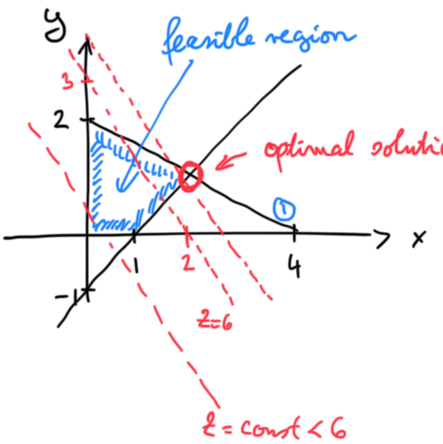
Graphical solution of linear programs

Example 1: maximize $z = 3x + 2y$

subject to
$$\begin{aligned} x + 2y &\le 4 \quad (1)\\ x - y &\le 1 \quad (2)\\ x, y &\ge 0 \end{aligned}$$

Line $z = 6 = 3x + 2y$



optimal solution, it's a "cornerpoint feasible (CPF) solution"

At the optimal corner point: $x + 2y = 4$, $x - y = 1$

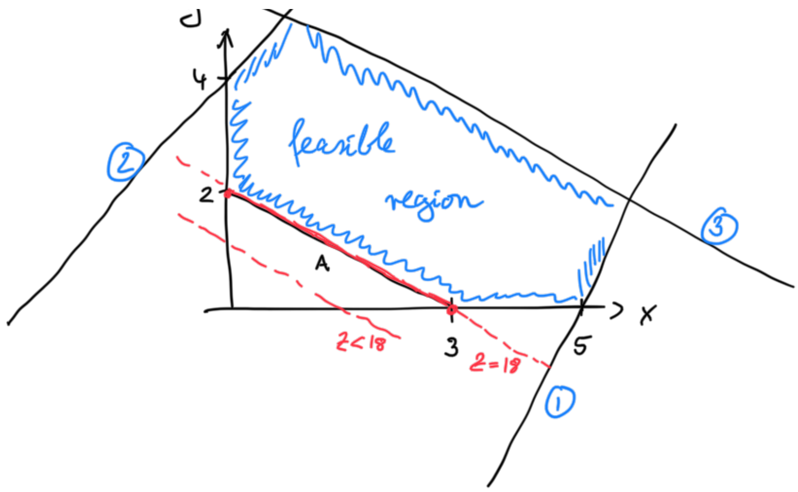
$$\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & -1 & 1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -3 & -3 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right)$$

Solution is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ with $z = 3 \cdot 2 + 2 \cdot 1 = 8$

Example 2: minimize $z = 6x + 9y$

subject to
$$\begin{aligned} 3x - y &\le 15 \quad (1)\\ -x + y &\le 4 \quad (2)\\ 2x + 5y &\le 27 \quad (3)\\ 2x + 3y &\ge 6 \\ x, y &\ge 0 \end{aligned}$$

$z = 18 = 6x + 3y$

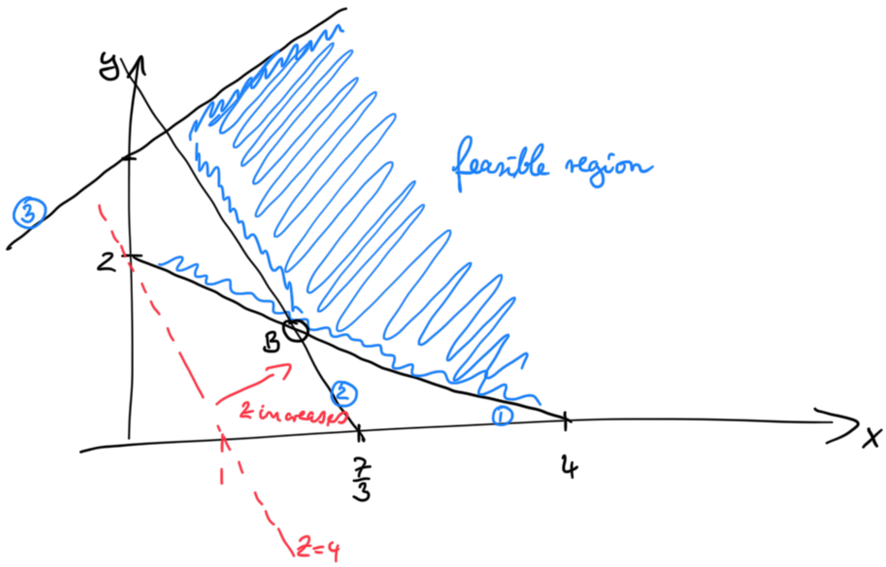


Any point on the edge A of the feasible region is an optimal solution

(infinitely many optimal solutions – the problem is degenerate)

Example 3: maximize $z = 4x + 2y$

subject to
$$x + 2y \geq 4 \quad (1)$$
$$3x + y \geq 7 \quad (2)$$
$$-x + 2y \leq 7 \quad (3)$$
$$x, y \geq 0$$



The feasible region is unbounded and z is increasing in the unbounded direction, so there are feasible solutions, but we cannot find an optimal solution.

Remark: The minimum of z exists and is taken at corner point B

There: $x + 2y = 4$
$3x + y = 7$

$$\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 3 & 1 & 7 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -5 & -5 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right)$$

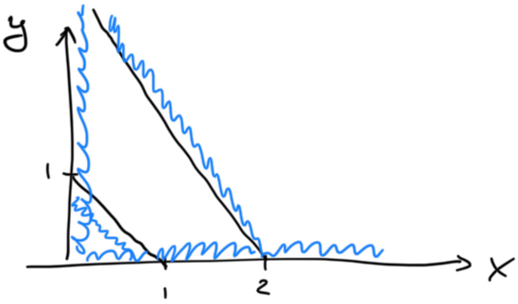
So $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ with $z = 4 \cdot 2 + 2 \cdot 1 = 10$

<u>Example 4:</u> maximize $z = 3x + 4y$

subject to $x + y \leq 1$

$2x + y \geq 4$

$x, y \geq 0$



Here: feasible region is empty

$\Rightarrow$ no feasible solutions

"the problem is over-constrained"

<u>Prototypical problems:</u>

① Activity analysis problem:

A: set of activities (or products)

R: set of resources (or production facilities)

w_{ij}: workload required from activity $i \in A$ on resource $j \in R$

c_j: available capacity of resource $j \in R$

p_i: profit from performing one unit of activity $i \in A$

Decision variables:

x_i: number of units of activity $i \in A$ to perform

$$\boxed{\begin{array}{lll} \text{maximize} & z = \sum_{i \in A} p_i x_i & \\ \text{subject to:} & \sum_{i \in A} w_{ij} x_i \leq c_j & \text{for all } j \in R \\ & x_i \geq 0 & \text{for all } i \in A \end{array}}$$

2. The diet problem

Given: $\mathcal{F}$: set of foods

N : set of nutrients

c_i : unit cost of food $i \in \mathcal{F}$

r_j : minimum requirements for nutrient $j \in N$

a_{ij} : amount of nutrient $j \in N$ from eating one unit of food $i \in \mathcal{F}$

Decision variables:

x_i : # of units of food $i \in \mathcal{F}$ to consume

$$\boxed{\begin{aligned} &\text{minimize} \quad && z = \sum_{i \in \mathcal{F}} c_i x_i && \\ &\text{subject to} \quad && \sum_{i \in \mathcal{F}} a_{ij} x_i \geq r_j && \text{for every } j \in N \\ & && x_i \geq 0 && \text{for } i \in \mathcal{F} \end{aligned}}$$