Midterm Exam Solutions

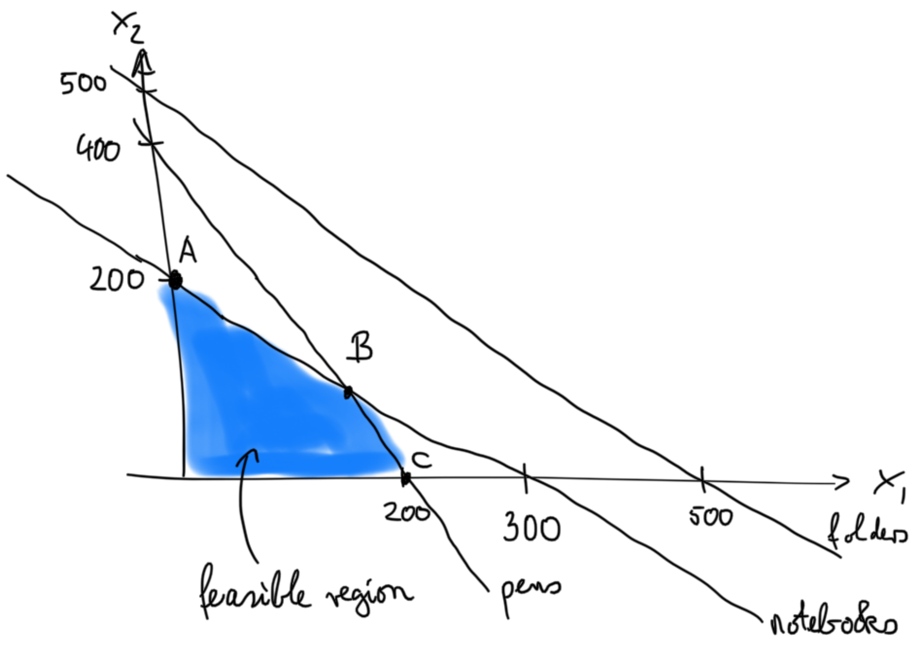
1. Let x_1 and x_2 denote the number of packages 1 resp. 2 to be put together.

Then the LP reads:

$$\text{maximize} \quad z = 6.5x_1 + 7x_2$$

$$\begin{aligned}
\text{subject to} \quad 2x_1 + 3x_2 &\le 600 \quad \text{(notebooks)} \\
1x_1 + 1x_2 &\le 500 \quad \text{(folders)} \\
2x_1 + 1x_2 &\le 400 \quad \text{(pens)} \\
x_1 \geq 0, \quad x_2 &\geq 0
\end{aligned}$$

Graphical solution:



Coordinates of point B, where pens and notebooks are binding:

$$\left(\begin{array}{cc|c} 2 & 3 & 600 \\ 2 & 1 & 400 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & 1 & 400 \\ 0 & 2 & 200 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & 0 & 300 \\ 0 & 1 & 100 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 0 & 150 \end{array}\right) \quad x_1 = 150, \quad x_2 = 100$$

$$\rightarrow \left(\begin{array}{cc|c} 0 & 1 & 100 \end{array} \right) \quad \Rightarrow \quad x_1 = 150, \quad x_2 = 100$$

Here, it's easiest to check z at A, B, C by direct insertion:

$$@A: \quad z = 0 + 7 \cdot 200 = 1400$$

$$@B: \quad z = 150 \cdot 6.5 + 100 \cdot 7 = 1675 \quad \Leftarrow \text{maximum here.}$$

$$@C: \quad z = 200 \cdot 6.5 = 1300$$

Alternatively, note that the slope of $z = \text{const}$ is close to -1, the slope of "notebooks" is $-\frac{2}{3} \gg -1$, the slope of "pens" is $-2 \ll -1$, which identifies B as the location of the maximum.

Solution via the simplex method (not required):

x_1	x_2	s_1	s_2	s_3	
2	3	1	0	0	600
1	1	0	1	0	500
2	1	0	0	1	400
-6.5	-7	0	0	0	0

= basic variables

Take x_1 as entering variable, R3 must become the pivot row for x_1:

x_1	x_2	s_1	s_2	s_3	
0	2	1	0	-1	200
0	$\frac{1}{2}$	0	1	$-\frac{1}{2}$	300
1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	200
0	$-\frac{15}{4}$	0	0	$\frac{13}{4}$	1300

Take x_2 as entering variable, R1 must become the pivot row for x_2:

x_1	x_2	s_1	s_2	s_3	
0	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	100
0	0	$-\frac{1}{4}$	1	$-\frac{1}{4}$	250
1	0	$-\frac{1}{4}$	0	$\frac{3}{4}$	150
0	0	$\frac{15}{8}$	0	$\frac{11}{8}$	1675

The method terminates here and we obtain the same result as before.

We can also read off straight away that at the optimal solution, there are 250 folders left $(S_2 = 250)$.

2. (a) minimize $-z = -x_1 - 2x_2$

subject to
$$-2x_1 + x_2 + x_3 + s_1 = 2$$
$$-x_1 + x_2 - x_3 + s_2 = 1$$
$$x_1, x_2, x_3, s_1, s_2 \geq 0$$

(b)

x_1	x_2	x_3	s_1	s_2	
-2	1	1	1	0	2
-1	1	-1	0	1	1
-1	-2	0	0	0	0

x_2 is entering, with pivot in R2:

x_1	x_2	x_3	s_1	s_2	
-1	0	2	1	-1	1
-1	1	-1	0	1	1
-3	0	-2	0	2	2

x_1 should be entering, but all its coefficients are negative

$\longrightarrow$ termination as an unbounded solution.

(c) This means that we can take x_1 arbitrary and move it onto the right hand side:

x_2	x_3	s_1	s_2	
0	2	1	-1	$1+x_1$
1	-1	0	1	$1+x_1$
0	-2	0	2	$2+3x_1$

So if we want $z = 2+3x_1 = 1000$, we can put

$$x_1 = \frac{998}{3} \quad \Rightarrow \quad x_2 = 1+x_1 = \frac{1001}{3} \qquad x_3 = 0$$

(Check: $z = \frac{998}{3} + 2\,\frac{1001}{3} = 1000$

$$-2x_1 + x_2 + x_3 = -\frac{1996}{3} + \frac{1001}{3} < 0 \leq 2$$

$$-x_1 + x_2 - x_3 = -\frac{998}{3} + \frac{1001}{3} = 1.\;)$$

(d) minimize $2y_1 + y_2$

subject to

$$-2y_1 - y_2 \geq 1 \qquad (*)$$

$$y_1 + y_2 \geq 2$$

$$y_1 - y_2 \geq 0$$

$$y_1, y_2 \geq 0$$

(e) We see immediately that $(*)$ cannot be satisfied if $y_1 \geq 0$ and $y_2 \geq 0$.

(recall that it is generally true by weak duality, that the dual problem

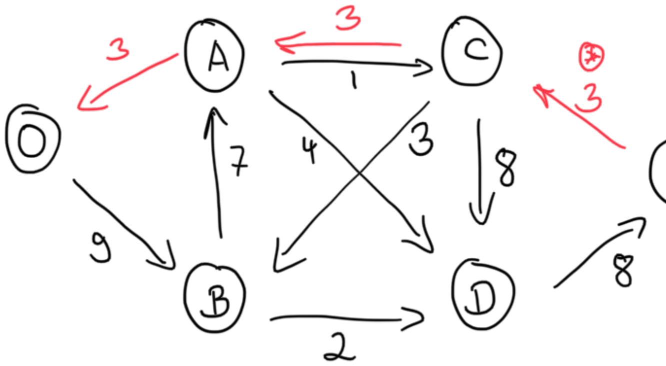
(Recall that [illegible]

is unfeasible if the primal problem is unbounded, and vice versa.)

3. Find max flow via augmenting path algorithm:

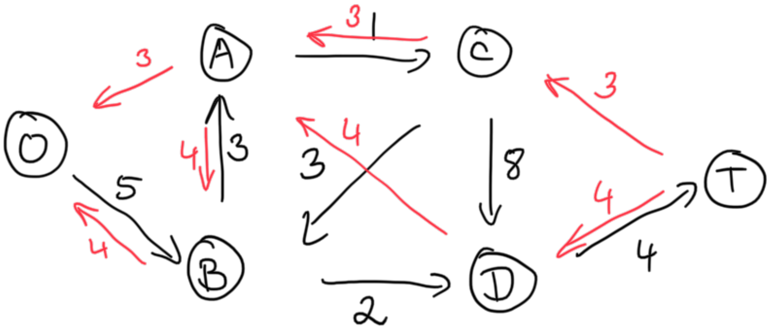
Stage 1: Path O – A – C – T with capacity 3 leaves

residual network

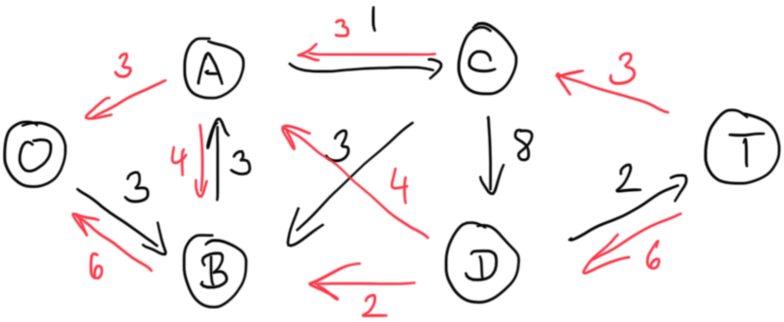


⊛ Add removed capacities as virtual capacities in the opposite direction. This allows to "undo" bad assignments of flows to arcs later, and is generally required for an optimal solution. In the example here, these virtual capacities are never used.

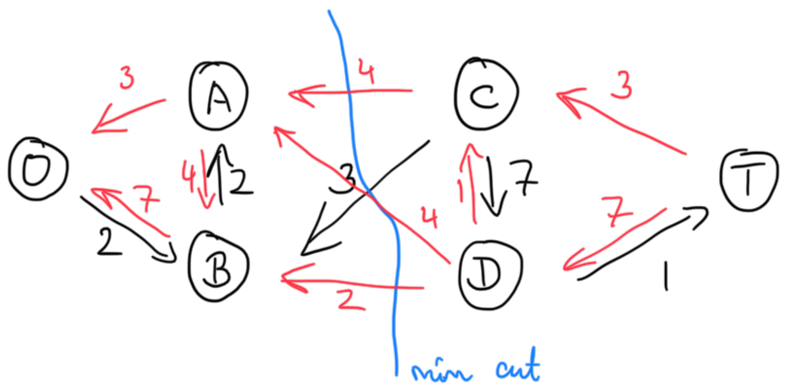
Stage 2: O – B – A – D – T with cap. 4 leaves:



Stage 3: O – B – D – T with cap. 2 leaves:



Stage 4: O – B – A – C – D – T with cap 1 leaves:



Now there is no augmenting path, so the min cut separates O, A, B on the left from C, D, T on the right.

The arcs on the min cut are (A,C) at cap. 4, (A,D) at cap. 4, and (B,D) at cap. 2. So the min cut (or max flow) has a total capacity of 10.

(Note: this can be written much more compactly, this level of detail was not required for a full score.)

4. Let A be the set of "arcs" or doors:

- $\{i,j\} \in A$ if there is a door from room i to room j
- order does not matter, thus set notation: $\{i,j\} = \{j,i\}$

Decision variable:

x_a : number of guards at door $a \in A$

LP:

minimize $\quad z = \sum_{a \in A} x_a \qquad$ (total number of guards)

subject to $\quad \sum_{a \in A} x_a \geq 1 \quad$ for every room $i \in R$

$$\text{s.t. } i \in a$$

$$x_a \geq 0$$

In general, we have to constrain the decision variables to be binary as this is not one of the network LP problems that has the integer solution property ! (Here, it will work with real decision variables as well, but that is by chance.)

<u>Data</u>: The arcs are $\{A,B\}$, $\{B,H\}$, $\{H,I\}$, $\{I,G\}$, $\{G,A\}$, $\{I,J\}$, $\{A,F\}$, $\{F,D\}$, $\{A,C\}$, $\{C,D\}$, $\{D,E\}$

5. (a)

$$\text{minimize} \quad \sum_{\substack{j \in J \\ s \in S \\ s > D_j}} a_{js} \cdot (s - D_j) \cdot P_j$$

$$\text{subject to} \quad \sum_{s \in S} a_{js} = 1 \quad \text{for all } j \in J$$

$$\sum_{j \in J} a_{js} = 1 \quad \text{for all } s \in S$$

$$a_{js} \geq 0$$

(b) This is an assignment problem that assigns jobs to a sequence of time slots. Each job fills out exactly one slot. Job j is due at the end of slot D_j. If that due date is missed $(s > D_j)$, a late penalty P_j is applied per slot over the due date ($s - D_j$ is the number of slots the job is late.) The task is to minimize the total penalty while case). The cost

still assigning all the jobs.

(c) The assignment problem has the integer solution property, so the solution will not change.