1. Suppose that for a certain data set, the semilog graph (base-10, logarithm on the y-axis) is a line through the points $(1, 1)$ and $(2, -3)$. Give an equation for y as a function of x.

(10)

If $\log_{10} y = mx + b$

then $y = 10^b \; 10^{mx}$.

Here: $1 = m \cdot 1 + b$

$-3 = m \cdot 2 + b$

$-4 = m \quad \Rightarrow \quad b = 5$

$\Rightarrow \quad y = 10^5 \; 10^{-4x}$.

2. Compute the derivative of the following functions.

(a) $f(x) = (1+x)^3$

(b) $f(x) = x^5 e^{-x}$

(c) $f(x) = \dfrac{ux}{vx+w}$ where u, v, and w are constants.

(5+5+5)

(a) $f'(x) = 3(1+x)^2 \cdot \underbrace{(1+x)'}_{=1} = 3(1+x)^2$

(b) $f'(x) = 5x^4 e^{-x} + x^5(-e^{-x})$

$= (5-x)x^4 e^{-x}$

(c) $f'(x) = \dfrac{u(vx+w) - v \cdot ux}{(vx+w)^2} = \dfrac{uw}{(vx+w)^2}$

3. The arctan function is a special function. You will find the following formula for its derivative in any table of mathematical functions:

$$(\arctan x)' = \frac{1}{1+x^2}.$$

Use this information to find the derivative of

$$f(x) = \arctan \frac{1}{x^2}.$$

(10)

$$f'(x) = \arctan'\left(\frac{1}{x^2}\right) \cdot \left(\frac{1}{x^2}\right)' \qquad \text{(chain rule)}$$

$$= \frac{1}{1+\left(\frac{1}{x^2}\right)^2} \cdot \left(-2x^{-3}\right)$$

$$= \frac{-2x}{x^4+1}$$

4. Find the derivative of

$$f(x) = \frac{1}{x}$$

by explicitly computing the limit of the difference quotient

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

(10)

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \frac{x - (x+h)}{x(x+h)}$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{-h}{x(x+h)}$$

$$= - \lim_{h \to 0} \frac{1}{x(x+h)}$$

$$= - \frac{1}{x^2}$$

5. Find the equation of the tangent line at $x = 0$ for the graph of the function

$$\ell(x) = \ln(1 + Kx) \qquad \Rightarrow \ell'(x) = \frac{K}{1+Kx}$$

where K is a given constant. (10)

We need to determine the coefficients of the equation for the tangent line

$$t(x) = mx + b$$

such that $t(0) = \ell(0)$ (*)

and $t'(0) = \ell'(0) \Rightarrow m = \frac{K}{1+K\cdot 0} = K$

Then, condition (*) reads

$$m \cdot 0 + b = \ln(1 + K \cdot 0)$$

$$\Rightarrow \quad b = \ln 1 = 0$$

$$\Rightarrow \boxed{t(x) = Kx}$$

6. Consider the function

$$f(x) = \frac{1}{4x} + \ln 4x$$

For which values of x is the function defined? Find the vertical and horizontal asymptotes (if any), find and classify all critical points, determine where the function is concave up or concave down, find all points of inflection, and sketch the graph.

(State the explicit y values for the special points you found only where they are easy to compute.) (5+5+5+5+5)

① ln is defined only for positive arguments, so need $\boxed{x>0}$

This excludes a zero denominator in the first term, too.

② $\lim_{x\to 0} f(x) = \infty$ (the log diverges more slowly than any power of x!)

$\Rightarrow$ vertical asymptote at $x=0$

$\lim_{x\to\infty} f(x) = \infty$, so no horizontal asymptote.

③ $f'(x) = \frac{1}{4}\left(-\frac{1}{x^2}\right) + \frac{4}{4x} = \frac{4x-1}{4x^2}$

critical points: $f'(x) = 0 \Rightarrow 4x-1=0 \Rightarrow x = \frac{1}{4}$

at $x=\frac{1}{4}$, f' changes sign from $-$ to $+$ $\Rightarrow$ local minimum.

The corresponding y-value is $f(\frac{1}{4}) = 1 + \ln 1 = 1$.

④ $$f''(x) = \frac{4(4x^2) - 8x(4x-1)}{16x^4}$$

$$= \frac{16x^2 - 32x^2 + 8x}{16x^4}$$

$$= \frac{-16x + 8}{16x^3}$$

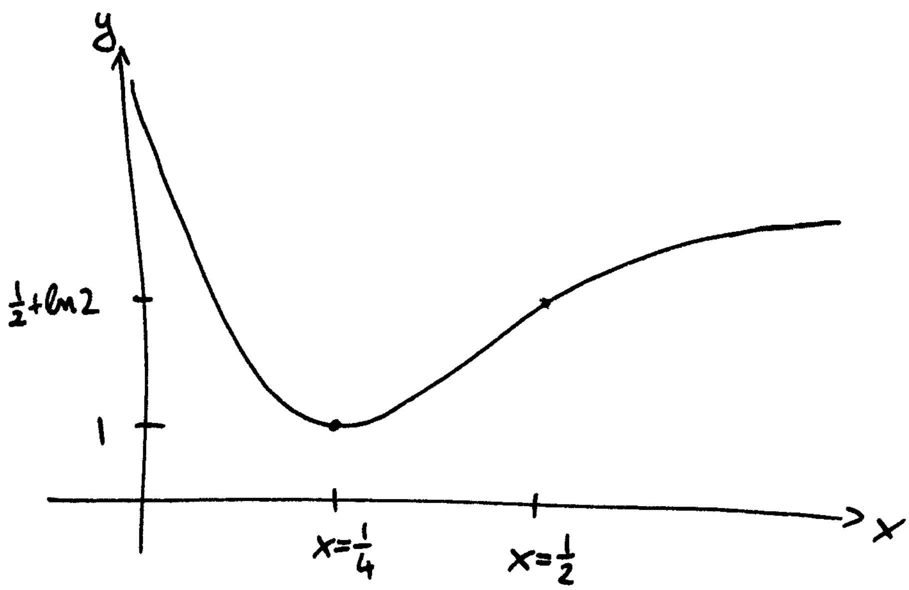
$f''(x) = 0 \implies -16x + 8 = 0 \implies x = \frac{1}{2}$ $\left(y = \frac{1}{2} + \ln\frac{1}{2}2\right)$

at $x = \frac{1}{2}$, f'' is changing sign $\implies$ point of inflection.

For $x < \frac{1}{2}$, $f''(x) > 0$, f concave up

$x > \frac{1}{2}$, $f''(x) < 0$, f concave down

⑤



7. You drive the 300 km distance from Berlin to Hamburg at an average speed of 100 km/h. Then you go on to Bremen, which is 100 km from Hamburg, where there is a traffic jam on the A1 so that you average speed drops down to 50 km/h on this part of your journey. What is your average speed over the entire distance? (10)

Berlin - Hamburg:

$$\text{travel time } t_1 = \frac{300 \text{ km}}{100 \frac{\text{km}}{\text{h}}} = 3 \text{ h}$$

Hamburg - Bremen:

$$\text{travel time } t_2 = \frac{100 \text{ km}}{50 \frac{\text{km}}{\text{h}}} = 2 \text{ h}$$

total travel time $t = t_1 + t_2 = 5 \text{ h}$

total distance $d = 300 \text{ km} + 100 \text{ km} = 400 \text{ km}$

$$\Rightarrow \text{average speed} = \frac{\text{total distance}}{\text{total time}} = \frac{400 \text{ km}}{5 \text{ h}} = 80 \frac{\text{km}}{\text{h}}$$