1. (a) Find the equation of the line through the points $(-3,-2)$ and $(1,1)$.
   (b) Find the equation of the quadratic polynomial $y = a x^2 + b x + c$ through the points $(0,0)$, $(5,0)$, and $(1,1)$.

(5+10)

(a) General form: $y = mx + b$

$$\Rightarrow \quad \begin{aligned} -2 &= m(-3) + b \\ 1 &= m \cdot 1 + b \\ \hline -3 &= m(-4) + 0 \end{aligned} \qquad \Rightarrow \quad m = \frac{3}{4}$$

$$\Rightarrow \quad 1 = \frac{3}{4} + b \qquad \Rightarrow \quad b = \frac{1}{4}$$

(b) Solution 1: We see that $x=0$ and $x=5$ are roots, so the polynomial must be of the form $y = d\,x\,(x-5)$. Using the last point $(1,1)$, we must have

$$1 = d \cdot 1 \cdot (1-5)$$

$$\Rightarrow \quad d = -\frac{1}{4} \qquad \text{so that } a = -\frac{1}{4} \text{ and } b = \frac{5}{4},\ c = 0.$$

Solution 2: (brute force)

$$0 = a \cdot 0 + b \cdot 0 + c \quad \Rightarrow \quad c = 0$$

$$\left.\begin{aligned} 0 &= a \cdot 5^2 + b \cdot 5 + 0 \\ 1 &= a \cdot 1^2 + b \cdot 1 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} 0 &= a \cdot 5 + b \\ 1 &= a + b \\ \hline -1 &= a \cdot 4 \end{aligned} \quad \Rightarrow a = -\frac{1}{4}$$

$$\Rightarrow \quad 1 = -\frac{1}{4} + b \quad \Rightarrow \quad b = \frac{5}{4}$$

2. Solve the following equations for x.

(a) $5^x = 25$

(b) $\log_{10}(10 \log_2 x) = 3$

(c) $e^{\frac{2 \ln x^s}{s}} = 1$

(5+5+5)

(a) $x = 2$ (by inspection ...)

(b) $10 \log_2 x = 1000 \Rightarrow \log_2 x = 100 \Rightarrow x = 2^{100}$

(c) $\frac{2}{s} \ln x^s = \ln x^{s \frac{2}{s}} = \ln x^2$ (if $s \neq 0$)

$\Rightarrow e^{\frac{2 \ln x^s}{s}} = e^{\ln x^2} = x^2$

So $x = 1$ is a solution. Note that $x = -1$ might be a solution, but the original expression is not well-defined for x negative except when s is a positive integer.

3. Suppose that for a certain data set, the doubly logarithmic (base-10) graph is a line through points $(1, 3)$ and $(2, -2)$. Give an equation for x as a function of y. (10)

A straight line on a doubly logarithmic plot corresponds to an allometric relationship of the form

$$y = a x^{b}$$

$$\Rightarrow \log_{10} y = \log_{10} a + b \log_{10} x$$

The data points are:

$$(\log_{10} x, \log_{10} y) = (1, 3)$$

$$(\log_{10} x, \log_{10} y) = (2, -2)$$

Thus, we need to solve the system of linear equations

$$3 = \log_{10} a + b \cdot 1 \qquad (*)$$

$$-) \quad -2 = \log_{10} a + b \cdot 2$$

$$5 = -b \qquad \Rightarrow b = -5$$

When we insert $b = -5$ into $(*)$, we obtain

$$3 = \log_{10} a - 5 \qquad \Rightarrow \log_{10} a = 8 \quad \Rightarrow a = 10^{8}$$

Altogether:

$$y = 10^{8} \cdot x^{-5}$$

4. The number of bacteria in milk grows exponentially, at least for some time. At bottling time, it is known that there are 10^6 bacteria per bottle, the next day, there are three times as many. The milk can be consumed with up to 10^9 bacteria per bottle.

   (a) Determine the shelf-life of bottled milk under these assumptions.

   (b) You are investigating the growth of bacteria in a sample of food and want to plot the number of bacteria vs. time. Which scaling function will you use on each of the coordinate axes, and why?

(10+5)

(a) The number of bacteria satisfies an expression of the form

$$B(t) = B_0 e^{\lambda t} \qquad \text{with } B_0 = 10^6$$

$$B(1) = 3B_0 \quad \Rightarrow \quad B_0 e^{\lambda} = 3B_0$$

$$\Rightarrow \quad e^{\lambda} = 3$$

$$\Rightarrow \quad \lambda = \ln 3$$

Now if s is the shelf-life, then $B(s) = 10^9$

$$\Rightarrow \quad B_0 e^{\lambda s} = 10^9$$

$$\Rightarrow \quad e^{\lambda s} = 1000$$

$$\Rightarrow \quad \lambda s = \ln 1000$$

$$\Rightarrow \quad s = \frac{\ln 1000}{\ln 3} \approx 6 \quad (\text{days})$$

(b) You would plot $\ln B$ (or $\log_{10} B$) vs. time. This would enable to determine the coefficients of an exponential relationship by fitting a straight line.

5. Compute the following limits.

(a) $\lim\limits_{z \to 10} \dfrac{\log_{10} z}{z}$

(b) $\lim\limits_{z \to 2} \dfrac{z^2 - 4}{z - 2}$

(c) $\lim\limits_{z \to \infty} \dfrac{e^z + z}{e^z}$

(5+5+5)

(a) The expression is continuous at $z = 10$, so

$$\lim_{z \to 10} \frac{\log_{10} z}{z} = \frac{\log_{10} 10}{10} = \frac{1}{10}$$

(b) $$\lim_{z \to 2} \frac{z^2 - 4}{z-2} = \lim_{z \to 2} \frac{(z-2)(z+2)}{z-2} = \lim_{z \to 2} z+2 = 4$$

(c) $$\lim_{z \to \infty} \frac{e^z + z}{e^z} = \lim_{z \to \infty} \frac{1 + z e^{-z}}{1} = 1$$

as $\lim\limits_{z \to \infty} z e^{-z} = 0$.

6. Determine whether $\ell(x)$ is continuous. If $\ell(x)$ has a discontinuity, state the type of discontinuity (removable discontinuity, jump discontinuity, vertical asymptote, or other).

(a) $\ell(x) = \frac{1}{x}$

(b) $\ell(x) = x \ln x^2$

(c) $\ell(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ x^2 & \text{for } x > 0 \end{cases}$

(5+5+5)

(a) ℓ has a vertical asymptote at $x=0$

(b) ℓ is not defined at $x=0$, but $\lim\limits_{x\to 0} x \ln x^2 = 0$,

so it's a removable discontinuity

(c) Since $\lim\limits_{\substack{x\to 0 \\ x<0}} 0 = 0$ and $\lim\limits_{\substack{x\to 0 \\ x>0}} x^2 = 0$,

Right-hand and left-hand limit coincide at $x=0$, so ℓ is continuous there.

Note: All the given functions are clearly continuous at all other points as compositions of continuous functions.

7. An elevator is driven by a motor which is either off or moves it with the constant speed of 1 m/s up or down.

The elevator is initially at the bottom of a building. At time $t = 40\,\mathrm{s}$, it visits the third floor at height $h = 12\,\mathrm{m}$, then at $t = 60\,\mathrm{s}$ the first floor at height $h = 4\,\mathrm{m}$.

(a) Is the height function $h(t)$ continuous? Why or why not?

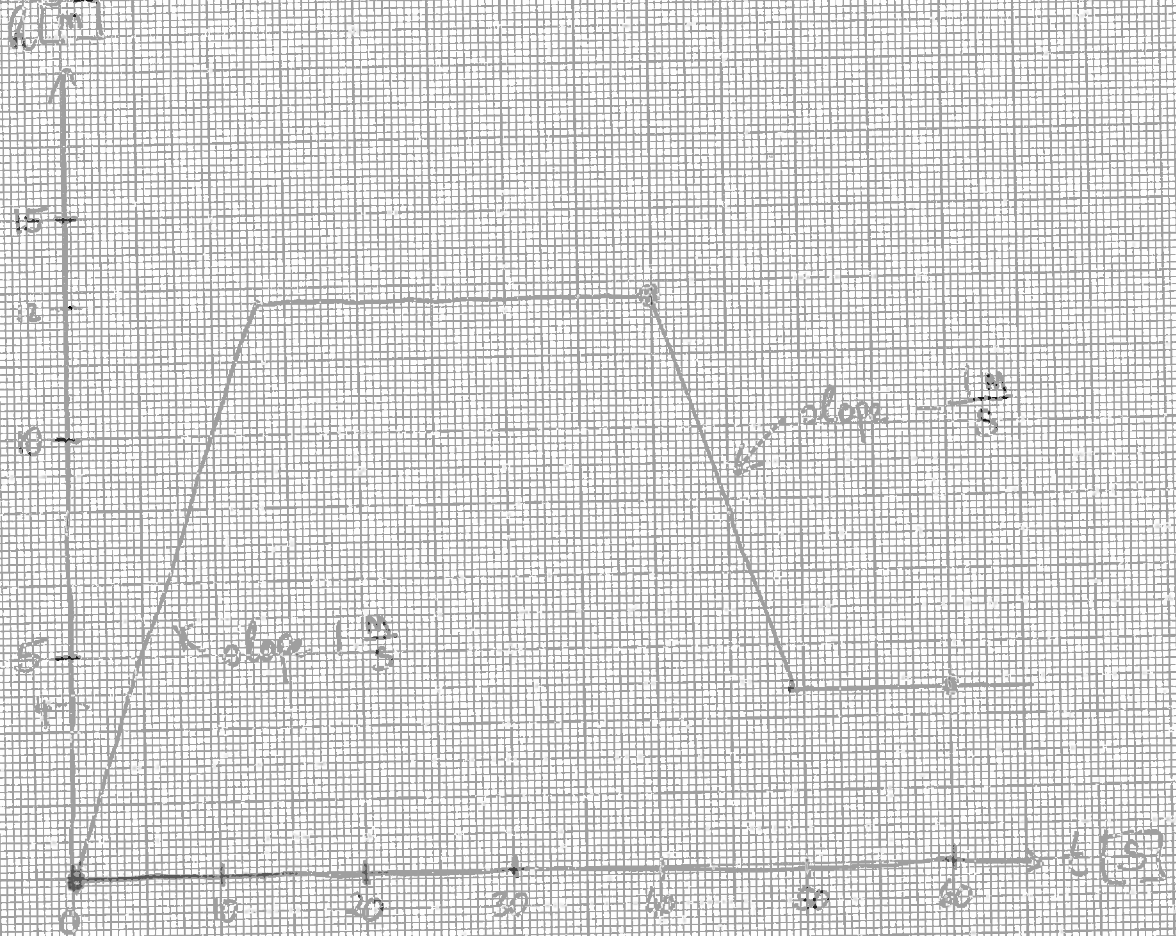
(b) Draw a possible height function $h(t)$ onto the graph paper provided. Label the coordinate axes carefully.

(Note: The answer is not unique!)

(5+10)

(a) Yes, an elevator cannot jump from one height to another, and at the timescale of the data given, the transitions are very relevant features of the mathematical description.

(b) p.t.o.


h [m]
15
12
10
5
4
slope 1 m/s
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