Derivatives Lab

Session 13

October 30, 2012

1. Approximate the Itō integral

$$I = \int_0^T X(t-)\,\mathrm{d}W(t) = \lim_{N\to\infty} \sum_{i=0}^{N-1} X(t_i)\,(W(t_{i+1}) - W(t_i))$$

and the Stratonovich integral

$$W = \int_0^T X(t)\,\mathrm{d}W(t) = \lim_{N\to\infty} \sum_{i=0}^{N-1} X\Big(\frac{t_{i+1}+t_i}{2}\Big)\,(W(t_{i+1}) - W(t_i))\,,$$

where $W(t)$ denotes standard Brownian motion, $t_i = i\,\Delta t$ with $\Delta t = T/N$, and we choose, as an example, $X(t) = W(t)$. Do they converge to the same value as $N \to \infty$?

2. It is known that the stochastic differential equation

$$\mathrm{d}S(t) = \mu\,S(t)\,\mathrm{d}t + \sigma\,S(t)\,\mathrm{d}W(t)\,, \qquad (*)$$
$$S(0) = S_0$$

is solved by geometric Brownian motion

$$S(t) = S_0\,\mathrm{e}^{(\mu-\sigma^2/2)\,t+\sigma\,W(t)}\,.$$

(a) Use the Euler–Maruyama method to solve (*) with $\mu = 2$ and $\sigma = 1$, and compare pathwise against the exact solution.

(b) Find the *strong order of convergence*, i.e., an exponent p such that

$$\mathbb{E}[|S_N - S(T)|] \le c\,\Delta t^p$$

where $S(T)$ denotes true geometric Brownian motion and S_N its Euler–Maruyama approximation at the final time T.

(c) Find the *weak order of convergence*, i.e., an exponent q such that

$$|\mathbb{E}[S_N] - \mathbb{E}[S(T)]| \le c\,\Delta t^q\,.$$