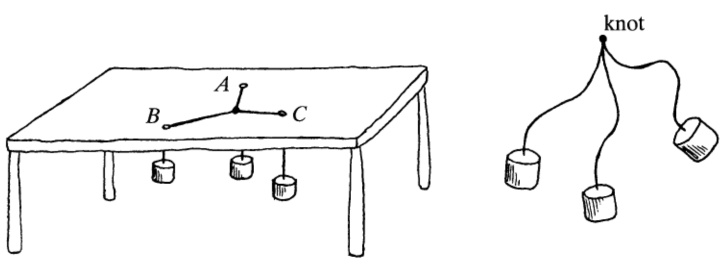
Perspectives of Mathematics I

Homework 2

due November 5, 2011

1. Consider a mechanical setup according the the following figure (from T.F. Tokieda, *Mechanical ideas in geometry*, The American Mathematical Monthly **105** (1998), 697–703) where the strings can slide freely through the holes.



   (a) Argue that, in equilibrium, the length of string above the table is minimal.

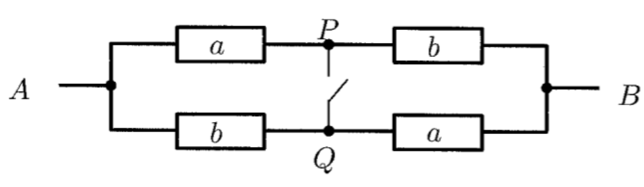
   (b) Argue that the angles around the knot are all 120°.

   Remark: the location of the knot is called the *short-center* or *brachycenter* of the triangle ABC.

2. (From M. Levi, *The Mathematical Mechanic*, Princeton University Press, 2009, p. 83.) Prove the inequality

$$\frac{1}{\frac{1}{a+b} + \frac{1}{c+d}} \geq \frac{1}{\frac{1}{a} + \frac{1}{c}} + \frac{1}{\frac{1}{b} + \frac{1}{d}}$$

   for positive real numbers a, b, c, d. Hint: consider the following circuit from Levi's book and change some resistances:



3. Consider the flow of a planar vector field (u, v) and denote points of the plane by (x, y).

   (a) A stick of infinitesimal length, initially parallel to the y-axis, is carried along by the flow. Show that its initial angular velocity is given by $-\partial u/\partial y$.

   (b) Conclude that the planar curl of the vector field, $\partial v/\partial x - \partial u/\partial y$, is twice the average angular velocity of the flow.

   (Without loss of generality, you may translate the point of interest into the origin.)

4. (a) Draw a careful sketch of the vector field the flow lines corresponding to the complex-valued function
$$f(z) = \frac{1}{z^2}\,.$$

   (b) What are flux and circulation about a small loop enclosing the origin? Argue by symmetry.

   (c) What are flux and circulation about a small loop enclosing the origin for the vector field corresponding to $f(z) = z^{-n}$ for integers $n \geq 3$? (You may argue by symmetry, or show a computation.)

5. Show, by looking at the flow of the vector field corresponding to the function
$$f(z) = \frac{\cot(\pi z)}{2\,z^4}$$
that
$$\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots\,.$$