Derivatives Lab

Session 22

December 5, 2011

Suppose S_i for $i = 0, \ldots, N$ denotes time series data which we believe behaves like geometric Brownian motion. Then estimates for the parameters μ and σ can be obtained in the following way.

Consider the log-returns

$$r_i = \ln S_{i+1} - \ln S_i \,,$$

then compute the sample mean

$$\bar{r} = \frac{1}{N} \sum_{i=0}^{N-1} r_i$$

and sample variance

$$\sigma_r^2 = \frac{1}{N-1} \sum_{i=0}^{N-1} (r_i - \bar{r})^2 \,.$$

Then the estimates for σ and μ are given by

$$\hat{\sigma} = \frac{\sigma_r}{\sqrt{\Delta t}} \qquad \text{and} \qquad \hat{\mu} = \frac{\bar{r}}{\Delta t} + \frac{\hat{\sigma}^2}{2} \,.$$

1. Generate a large number of sample geometric Brownian paths with fixed μ and σ. For each, compute the estimates $\hat{\sigma}$ and $\hat{\mu}$.

   (a) Draw a histogram (command `hist`) for the values for the distribution of $\hat{\sigma}$ and $\hat{\mu}$.

   (b) It is known that the variance of the estimate for σ is approximately

   $$\operatorname{Var}[\hat{\sigma}] = \frac{\hat{\sigma}^2}{2N} \,.$$

   Does your statistics from part (a) reproduce this result?

   (c) What is the variance of $\hat{\mu}$? Is it large or small?

2. Modify your code in such a way that the number of points on the geometric Brownian path is $N = 2^k + 1$ so that the number of log-returns is a power of two. Now repeat the estimation of σ for a single geometric path over large time steps of length $2^i \, \Delta t$ where $i = 0, \dots, k-1$ and Δt is the time step of the original geometric Brownian path. Plot the $\hat{\sigma}$ vs. the log of the number of sample points (`semilogx`).

   How does this result change if

   (a) you add Gaussian noise to the geometric Brownian motion;

   (b) you add a high frequency periodic perturbation?

3. Perform a QQ-plot vs. the normal distribution for each of the three cases considered above.

4. Plot the autocorrelation function for each of the three cases considered above.