Engineering and Science Mathematics I

Advanced Placement Exam

September 8, 2010

Last Name: ______________________________

First Name: ______________________________

Signature: ______________________________

Some trigonometric identities:

$$\sin^2\theta + \cos^2\theta = 1 \qquad \sin 2x = 2\,\sin x\,\cos x$$

$$1 + \tan^2\theta = \sec^2\theta \qquad \cos 2x = 2\,\cos^2 x - 1 = 1 - 2\,\sin^2 x$$

$$1 + \cot^2\theta = \csc^2\theta \qquad \tan 2x = \frac{2\,\tan x}{1 - \tan^2 x}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$e^{i\theta} = \cos\theta + i\,\sin\theta$$

Useful integrals:

$$\int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C$$

$$\int \frac{du}{1+u^2} = \arctan u + C = -\operatorname{arccot} u + C'$$

$$\int \frac{du}{u\sqrt{u^2-1}} = \operatorname{arcsec}|u| + C$$

$$\int \sec u\,du = \ln\left|\sec u + \tan u\right| + C$$

1. Compute

   (a) $\lim_{n\to\infty} 1 + \frac{1}{n} + \frac{1}{n^2}$

   (b) $\lim_{a\to -1} \frac{a^2+4a+3}{a^2-2a-3}$

   (c) $\lim_{\theta\to 0} \frac{\theta - \sin\theta}{\theta^3}$

(5+5+5)

2. Consider the function

$$f(x) = \begin{cases} 0 & \text{for } x \le 0 \\ e^{-1/x} & \text{for } x > 0\,. \end{cases}$$

   (a) Is f continuous at $x = 0$? Explain.

   (b) Is f differentiable at $x = 0$? If so, find $f'(0)$.

(5+5)

3. Consider the graphs of the functions

$$f(x) = m\,x$$

   and

$$g(x) = \arctan x\,.$$

   For which values of m do they intersect in (a) one, (b) two, and (c) three points? (10)

4. Show that the square is the rectangle of maximal area for a given perimeter. (10)

5. For each $a > 0$, consider the function

$$f(p) = p\,a^{1/p}\,.$$

   Find all minima and maxima, points of inflection, and vertical and horizontal asymptotes on the interval $p \in (0, \infty)$. (10)

6. Find the indefinite integral

$$\int \cos^2\theta \, d\theta$$

   by

   (a) using trigonometric identities, and

(b) writing the expression in terms of complex exponentials. You should

(c) verify that the results from (a) and (b) are compatible.

(5+5+5)

7. Find the indefinite integrals

(a) $\displaystyle\int \frac{1}{1+e^x}\,dx$

(b) $\displaystyle\int \frac{4x-2}{x^3-x}\,dx$

(5+5)

8. Compute the Taylor series about the point $x = 0$ of

(a) $f(x) = (x+2)(x-2)$,

(b) $g(x) = \ln(1+x)$,

(c) $h(x) = \ln(x+1) + \ln(1/(x+1))$.

(5+5+5)

9. (a) Does the infinite series

$$\sum_{n=0}^{\infty} \frac{n}{n^2+1}$$

converge?

(b) Show that

$$\sum_{n=0}^{\infty} \left(\frac{\sin n}{2}\right)^n \leq 2$$

(5+5)

10. *Note:* In parts (a) and (b) of the following problem you may leave logarithms and exponential functions unevaluated.

The concentration $c(t)$ of a chemical catalyst decays at a rate $\lambda\, c(t)$, where $\lambda = 1\,\mathrm{s}^{-1}$.

(a) If the initial concentration of the catalyst is $c(0) = 10\,\mathrm{g/l}$, what is the concentration after $2\,\mathrm{s}$?

(b) How long does it take until half of the catalyst has decayed?

(c) Suppose the catalyst is added to a reactant with concentration $r(t)$ which undergoes a chemical reaction at rate $\nu\, c(t)\, r(t)$, where ν is some given constant. Show that the fraction of the reactant which remains after all chemical activity has subsided is given by the expression $e^{-\nu c(0)/\lambda}$.

(5+5+5)