1. Find all points (x, y) on the graph of $x^2 - xy + y^2 = 27$ where the tangent line is horizontal. (10)

Use implicit differentiation with $y = y(x)$:

$$2x - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

When the tangent line is horizontal, $\frac{dy}{dx} = 0$, i.e.

$$2x - y = 0 \quad \Rightarrow \quad y = 2x$$

Then

$$x^2 - 2x^2 + 4x^2 = 27$$

$$\Rightarrow \quad 3x^2 = 27$$

$$\Rightarrow \quad x^2 = 9$$

$$\Rightarrow \quad x = \pm 3$$

So the points are $(3, 6)$ and $(-3, -6)$.

2. Compute the following indefinite integrals $\quad U = \sin x \Rightarrow dU = \cos x \, dx$

(a) $\int \frac{\cos x}{\sqrt{4-\sin^2 x}} \, dx = \int \frac{dU}{\sqrt{4-U^2}}$

(b) $\int x^3 e^{-x^2} \, dx$

(c) $\int \frac{4x^2+x+1}{4x^3+x} \, dx$

(d) $\int \sin x \sec x \, dx$

(e) $\int x (\ln x)^3 \, dx$

(10+10+10+10+10)

(a) $U = 2 \sin\theta \qquad dU = 2\cos\theta \, d\theta$

$$\Rightarrow \int \frac{dU}{\sqrt{4-U^2}} = \int \frac{2\cos\theta \, d\theta}{\sqrt{4-4\sin^2\theta}} = \int d\theta = \theta + C$$

$$= \arcsin \frac{U}{2} + C = \arcsin \frac{\sin x}{2} + C$$

(b) $t = -x^2 \Rightarrow dt = -2x \, dx$

$$\Rightarrow \int x^3 e^{-x^2} dx = \frac{1}{2}\int t e^t \, dt = \frac{1}{2} t e^t - \frac{1}{2}\int e^t dt$$

$$= \frac{1}{2}(t-1)e^t + C$$

$$= -\frac{1}{2}(x^2+1)e^{-x^2} + C$$

(c) $$\frac{4x^2+x+1}{(4x^2+1)x} = \frac{A+Bx}{4x^2+1} + \frac{C}{x}\ 3 = \frac{Ax + Bx^2 + C(4x^2+1)}{x(4x^2+1)}$$

$A = 1, \ C = 1, \ B = 0$

$$\Rightarrow \int \frac{4x^2+x+1}{4x^3+x}\,dx = \int \frac{1}{4x^2+1}\,dx + \int \frac{1}{x}\,dx$$

$$= \frac{1}{2}\arctan 2x + \ln|x| + C$$

(d) $$\int \sin x \sec x \, dx = -\int \frac{du}{u}$$

$$u = \cos x$$
$$du = -\sin x \, dx$$

$$= -\ln|\cos x| + C$$

(e) $$\int x(\ln x)^3\,dx = \frac{1}{2}x^2(\ln x)^3 - \int \frac{1}{2}x^2\, 3(\ln x)^2 \frac{1}{x}\,dx$$

$$= \frac{1}{2}x^2(\ln x)^3 - \frac{3}{2}\underbrace{\int x(\ln x)^2\,dx}_{= \frac{1}{2}x^2(\ln x)^2 - \int \frac{1}{2}x^2 2\ln x \frac{1}{x}\,dx}$$

$$= \frac{1}{2}x^2(\ln x)^3 - \frac{3}{4}x^2(\ln x)^2 + \frac{3}{2}\underbrace{\int x\ln x\,dx}_{= \frac{1}{2}x^2\ln x - \frac{1}{2}\int x^2 \frac{1}{x}\,dx}$$

$$= \frac{1}{2}x^2(\ln x)^3 - \frac{3}{4}x^2(\ln x)^2 + \frac{3}{4}x^2\ln x - \frac{3}{8}x^2 + C$$

3. Compute the average value of the function $f(x) = \tan x$ on the interval $[-1, 1]$. (5)

$\tan x$ is an odd function, i.e. $\tan(-x) = -\tan x$

Hence, $\int_{-1}^{1} \tan x \, dx = 0$

(the contribution to the integral on $[0,1]$ and $[-1,0]$ are equal, but with opposite sign.)

$$\Rightarrow \quad \bar{f} = \frac{1}{1-(-1)} \int_{-1}^{1} \tan x \, dx = 0.$$

4. Does the improper integral

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx$$

converge? If so, compute its value. (10)

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{R\to\infty} \int_1^R x^{-\frac{1}{2}}\,dx$$

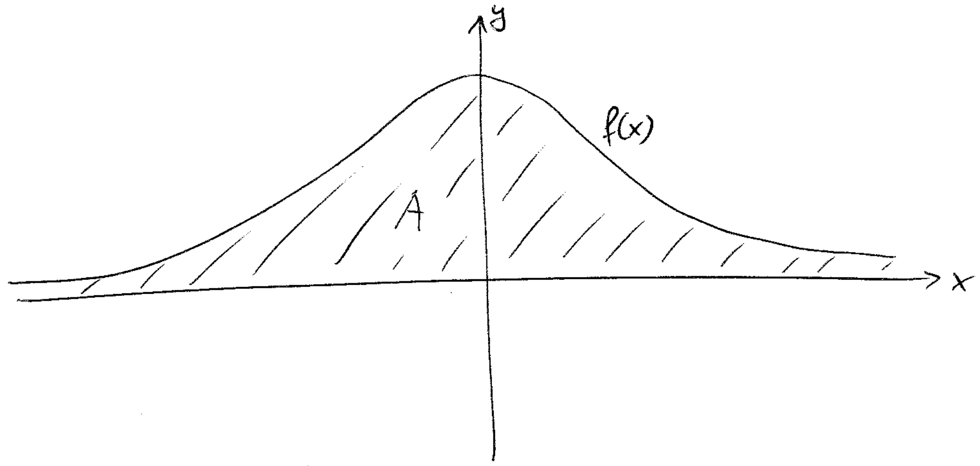
$$= \lim_{R\to\infty} 2x^{\frac{1}{2}}\Big|_1^R = \infty$$

$\Rightarrow$ The integral diverges.

5. Compute the area between the x-axis and the graph of

$$f(x) = \frac{1}{4+x^2}.$$

(10)

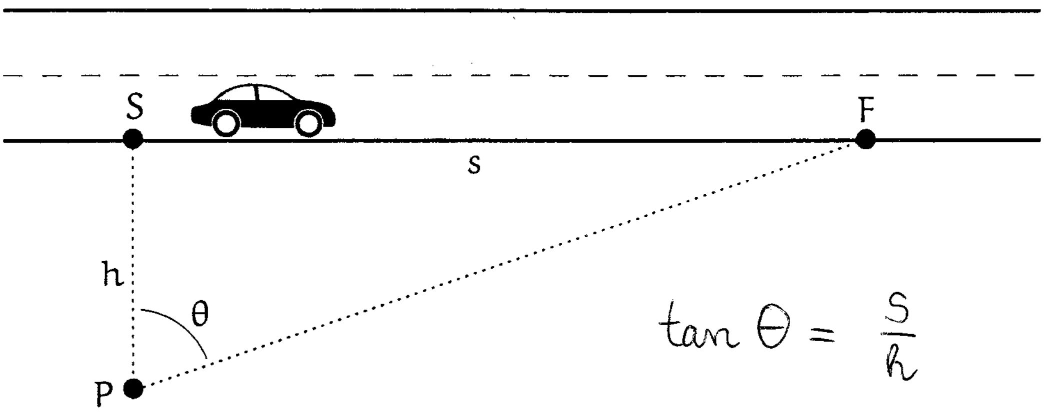


$$A = \int_{-\infty}^{\infty} \frac{1}{4+x^2}\,dx \qquad x = 2z \Rightarrow dx = 2dz$$

$$= \int_{-\infty}^{\infty} \frac{2\,dz}{4+4z^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dz}{1+z^2} = \frac{1}{2} \arctan z \Big|_{-\infty}^{+\infty}$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{2}$$

6. A policeman is positioned at point P alongside a straight stretch of road to catch speeding cars. He employs the following method: When a car passes the point S closest to him, he starts his stopwatch. He then looks through a spotting scope, mounted at an angle θ toward the road. When the car comes into view at point F, he stops the clock. The setting is shown in this figure:



$\tan\theta = \frac{s}{h}$

Suppose that the distance of the policeman to the road is h, that the angular resolution of the spotting scope is $\Delta\theta$ (in other words, $\Delta\theta$ is the expected measurement error of the angle θ), and that all other measurement errors are negligible.

(a) Use linear approximation to derive a formula for the expected error Δs in the computed traveled distance s.

(b) Then derive a formula for the expected error Δv in the computed velocity v of the car.

(c) What angle θ should the policeman use to minimize the error in the velocity?

Hint: In part (c) you should find that the optimal angle is independent of the concrete numerical values of h, v, and $\Delta\theta$.

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(a) $s = h\tan\theta \Rightarrow \frac{ds}{d\theta} = h\sec^2\theta \Rightarrow \Delta s \approx h\sec^2\theta\,\Delta\theta$

(b) $v = \frac{s}{t} \Rightarrow \Delta v \approx \frac{\Delta s}{t} \approx \frac{h\sec^2\theta\,\Delta\theta}{\frac{h\tan\theta}{v}} = v\,\frac{1}{\sin\theta\cos\theta}\,\Delta\theta$

(c) By symmetry, $\frac{1}{\sin\theta\cos\theta}$ has a minimum at $\theta = \frac{\pi}{4}$, so this is the optimal angle.

(Alternatively use that $\frac{1}{\sin\theta\cos\theta} = \frac{2}{\sin 2\theta}$ and find critical points.)