1. Compute the following limits provided they exist.

   (a) $\lim_{c \to 1} \frac{|c-1|}{c-1}$

   (b) $\lim_{c \to 1} \frac{c^3 - 3c^2 + 3c - 1}{c^2 - 2c + 1}$

   (c) $\lim_{c \to 0} \sin c \cos \frac{1}{c}$

(5+5+5)

(a) $\lim_{c \to 1^+} \frac{|c-1|}{c-1} = 1$, $\lim_{c \to 1^-} \frac{|c-1|}{c-1} = -1$

so limit as $c \to 1$ does not exist.

(b) $\frac{c^3 - 3c^2 + 3c - 1}{c^2 - 2c + 1} = \frac{(c-1)^3}{(c-1)^2} = c - 1 \xrightarrow[c \to 1]{} 0$

so limit as $c \to 1$ is zero

(c) $\lim_{c \to 0} \sin c \cos \frac{1}{c} \leq \lim_{c \to 0} \sin c = 0$

$\lim_{c \to 0} \sin c \cos \frac{1}{c} \geq \lim_{c \to 0} \sin c \, (-1) = 0$

so limit $\lim_{c \to 0} \sin c \cos \frac{1}{c} = 0$.

2. The function

$$\phi(x) = x \ln|x|$$

is not defined at $x = 0$.

(a) Can you assign a value to $\phi(0)$ in such a way to make ϕ continuous?

(b) Can you assign a value to $\phi'(0)$ in such a way to make ϕ' continuous?

(5+5)

(a) We know from class that $\lim_{x\to 0^+} x \ln x = 0$,

so $\lim_{x\to 0} x \ln|x| = 0$ and we can define $\phi(0) = 0$

(b) $\phi'(x) = \ln x + x\frac{1}{x} = \ln x + 1$ for $x > 0$.

So $\lim_{x\to 0^+} \phi'(x)$ does not exist

$\Rightarrow$ there is no way to assign $\phi'(0)$ as to make ϕ' continuous.

3. Compute the derivative of $k(x) = \sqrt{x}$ by

(a) using the definition of the derivative as the limit of a difference quotient,

(b) using the power rule of differentiation.

Confirm that the two results coincide. (5+10)

(a) $$k'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2} \frac{1}{\sqrt{x}}$$

(b) $$k(x) = x^{+\frac{1}{2}}$$

$$\Rightarrow k'(x) = \frac{1}{2} x^{-\frac{1}{2}}$$

So the answers coincide.

4. Consider the function

$$f(x) = -x\,e^x .$$

Find horizontal and vertical asymptotes, local minima, local maxima, and inflection points of f. Identify the regions where the graph of f is concave upward or concave downward. Finally, sketch the graph into the coordinate system provided. (10)

$\lim_{x \to \infty} f(x) = -\infty$, $\lim_{x \to -\infty} f(x) = 0$ by properties of exp.

$\Rightarrow$ horizontal asymptote $y = 0$ as $x \to -\infty$

critical points: $f'(x) = -e^x - x\,e^x = -(1+x)e^x$

$\Rightarrow$ critical point at $x = -1$ where f' changes from positive to negative, so the critical point corresponds to a maximum at

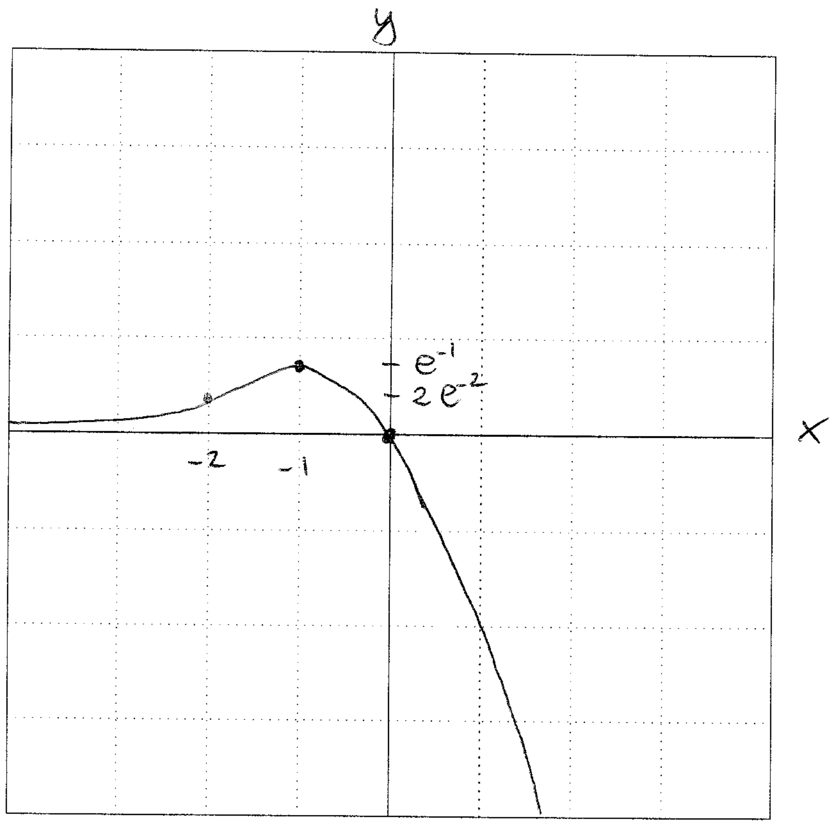
$x = -1$, $y = f(-1) = e^{-1} = \frac{1}{e}$

points of inflection: $f''(x) = -e^x - (1+x)e^x = -(2+x)e^x$

$f''(x) = 0$ when $x = -2$, there f'' changes from positive to negative, so there is a point of inflection at

$x = -2$, $y = f(-2) = 2e^{-2} = \frac{2}{e^2}$

To the left of $x = -2$, the graph is concave up, to the right it is concave down.



y
x
-2
-1
e^{-1}
2e^{-2}

5. A commuter train carries 800 passengers each day at a ticket price of €1.00 per person. Market research reveals that for each 5 cent increase/decrease of the price 50 fewer/more people would take the train. What fare should be charged to maximize revenue? (10)

Let x be the number of units of increase or decrease, i.e

price $P = 1 + x\frac{5}{100}$

passengers $N = 800 - x \cdot 50$

$\Rightarrow$ revenue $R = P \cdot N = \left(1 + x\frac{5}{100}\right)(800 - 50x)$

$$R'(x) = \frac{5}{100}(800 - 50x) + \left(1 + \frac{5}{100}x\right)(-50)$$

$\Rightarrow$ critical point at $\frac{5}{100}(800 - 50x) + \left(1 + \frac{5}{100}x\right)(-50) = 0$

$$\Rightarrow \quad 5(800 - 50x) = 50(100 + 5x)$$

$$\Rightarrow \quad 80 - 5x = 100 + 5x$$

$$\Rightarrow \quad 10x = -20$$

$$\Rightarrow \quad x = -2$$

Since R is quadratic with negative leading coefficient, this corresponds to a maximum. So the optimal price is

$$P_{opt} = 1 - 2\frac{5}{100} = \frac{90}{100} = 90 \text{ cent}$$