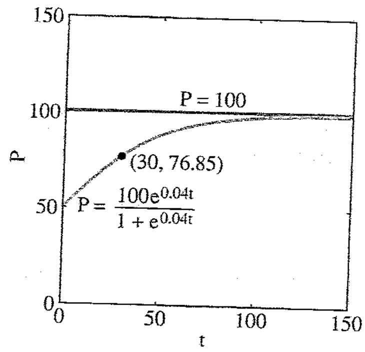


**Fig. 9.5.5** Graph of the population function in Eq. (12). It is apparent that $P(t) \to 100$ as $t \to +\infty$. Thus everyone eventually hears the rumor. Can you verify this directly by taking the limit in Eq. (12) as $t \to +\infty$?

$$\ln \frac{P}{100-P} = (0.04)t, \quad \text{so} \quad \frac{P}{100-P} = e^{(0.04)t}.$$

We readily solve this last equation for the solution

$$P(t) = \frac{100e^{(0.04)t}}{1+e^{(0.04)t}}. \tag{12}$$

Hence the number of people who have heard the rumor after 30 days is $P(30) \approx 76.85$ thousand people. See Fig. 9.5.5.

The method of Example 7 can be used to solve any differential equation of the form

$$\frac{dx}{dt} = k(x-a)(x-b), \tag{13}$$

where $a$, $b$, and $k$ are constants. As Problems 63 through 68 indicate, this differential equation serves as a mathematical model for a wide variety of natural phenomena.

## 9.5 PROBLEMS

Find the integrals in Problems 1 through 36.

**1.** $\displaystyle\int \frac{x^2}{x+1}\,dx$

**2.** $\displaystyle\int \frac{x^3}{2x-1}\,dx$

**3.** $\displaystyle\int \frac{1}{x^2-3x}\,dx$

**4.** $\displaystyle\int \frac{x}{x^2+4x}\,dx$

**5.** $\displaystyle\int \frac{1}{x^2+x-6}\,dx$

**6.** $\displaystyle\int \frac{x^3}{x^2+x-6}\,dx$

**7.** $\displaystyle\int \frac{1}{x^3+4x}\,dx$

**8.** $\displaystyle\int \frac{1}{(x+1)(x^2+1)}\,dx$

**9.** $\displaystyle\int \frac{x^4}{x^2+4}\,dx$

**10.** $\displaystyle\int \frac{1}{(x^2+1)(x^2+4)}\,dx$

**11.** $\displaystyle\int \frac{x-1}{x+1}\,dx$

**12.** $\displaystyle\int \frac{2x^3-1}{x^2+1}\,dx$

**13.** $\displaystyle\int \frac{x^2+2x}{(x+1)^2}\,dx$

**14.** $\displaystyle\int \frac{2x-4}{x^2-x}\,dx$

**15.** $\displaystyle\int \frac{1}{x^2-4}\,dx$

**16.** $\displaystyle\int \frac{x^4}{x^2+4x+4}\,dx$

**17.** $\displaystyle\int \frac{x+10}{2x^2+5x-3}\,dx$

**18.** $\displaystyle\int \frac{x+1}{x^3-x^2}\,dx$

**19.** $\displaystyle\int \frac{x^2+1}{x^3+2x^2+x}\,dx$

**20.** $\displaystyle\int \frac{x^2+x}{x^3-x^2-2x}\,dx$

**21.** $\displaystyle\int \frac{4x^3-7x}{x^4-5x^2+4}\,dx$

**22.** $\displaystyle\int \frac{2x^2+3}{x^4-2x^2+1}\,dx$

**23.** $\displaystyle\int \frac{x^2}{(x+2)^3}\,dx$

**24.** $\displaystyle\int \frac{x^2+x}{(x^2-4)(x+4)}\,dx$

**25.** $\displaystyle\int \frac{1}{x^3+x}\,dx$

**26.** $\displaystyle\int \frac{6x^3-18x}{(x^2-1)(x^2-4)}\,dx$

**27.** $\displaystyle\int \frac{x+4}{x^3+4x}\,dx$

**28.** $\displaystyle\int \frac{4x^4+x+1}{x^5+x^4}\,dx$

**29.** $\displaystyle\int \frac{x}{(x+1)(x^2+1)}\,dx$

**30.** $\displaystyle\int \frac{x^2+2}{(x^2+1)^2}\,dx$

**31.** $\displaystyle\int \frac{x^2-10}{2x^4+9x^2+4}\,dx$

**32.** $\displaystyle\int \frac{x^2}{x^4-1}\,dx$

**33.** $\displaystyle\int \frac{x^3+x^2+2x+3}{x^4+5x^2+6}\,dx$

**34.** $\displaystyle\int \frac{x^2+4}{(x^2+1)^2(x^2+2)}\,dx$

**35.** $\displaystyle\int \frac{x^4+3x^2-4x+5}{(x-1)^2(x^2+1)}\,dx$

**36.** $\displaystyle\int \frac{2x^3+5x^2-x+3}{(x^2+x-2)^2}\,dx$

In Problems 37 through 40, make a preliminary substitution before using the method of partial fractions.

**37.** $\displaystyle\int \frac{e^{4t}}{(e^{2t}-1)^3}\,dt$

**38.** $\displaystyle\int \frac{\cos\theta}{\sin^2\theta-\sin\theta-6}\,d\theta$

**39.** $\displaystyle\int \frac{1+\ln t}{t(3+2\ln t)^2}\,dt$

**40.** $\displaystyle\int \frac{\sec^2 t}{\tan^3 t+\tan^2 t}\,dt$

In Problems 41 through 44, find the area of the region $R$ between the curve and the $x$-axis over the given interval.

**41.** $y = \dfrac{x-9}{x^2-3x}$, $1 \leqq x \leqq 2$

**42.** $y = \dfrac{x+5}{3+2x-x^2}$, $0 \leqq x \leqq 2$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \int \frac{\sinh\theta}{\sinh\theta}\,d\theta = \int 1\,d\theta = \theta + C = \cosh^{-1}x + C.$$

The two results appear to differ, but Eq. (35) in Section 8.5 shows that they are equivalent.

## 9.6 PROBLEMS

Use trigonometric substitutions to evaluate the integrals in Problems 1 through 36.

1. $\displaystyle\int \frac{1}{\sqrt{16-x^2}}\,dx$

2. $\displaystyle\int \frac{1}{\sqrt{4-9x^2}}\,dx$

3. $\displaystyle\int \frac{1}{x^2\sqrt{4-x^2}}\,dx$

4. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-25}}\,dx$

5. $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$

6. $\displaystyle\int \frac{x^2}{\sqrt{9-4x^2}}\,dx$

7. $\displaystyle\int \frac{1}{(9-16x^2)^{3/2}}\,dx$

8. $\displaystyle\int \frac{1}{(25+16x^2)^{3/2}}\,dx$

9. $\displaystyle\int \frac{\sqrt{x^2-1}}{x^2}\,dx$

10. $\displaystyle\int x^3\sqrt{4-x^2}\,dx$

11. $\displaystyle\int x^3\sqrt{9+4x^2}\,dx$

12. $\displaystyle\int \frac{x^3}{\sqrt{x^2+25}}\,dx$

13. $\displaystyle\int \frac{\sqrt{1-4x^2}}{x}\,dx$

14. $\displaystyle\int \frac{1}{\sqrt{1+x^2}}\,dx$

15. $\displaystyle\int \frac{1}{\sqrt{9+4x^2}}\,dx$

16. $\displaystyle\int \sqrt{1+4x^2}\,dx$

17. $\displaystyle\int \frac{x^2}{\sqrt{25-x^2}}\,dx$

18. $\displaystyle\int \frac{x^3}{\sqrt{25-x^2}}\,dx$

19. $\displaystyle\int \frac{x^2}{\sqrt{1+x^2}}\,dx$

20. $\displaystyle\int \frac{x^3}{\sqrt{1+x^2}}\,dx$

21. $\displaystyle\int \frac{x^2}{\sqrt{4+9x^2}}\,dx$

22. $\displaystyle\int (1-x^2)^{3/2}\,dx$

23. $\displaystyle\int \frac{1}{(1+x^2)^{3/2}}\,dx$

24. $\displaystyle\int \frac{1}{(4-x^2)^2}\,dx$

25. $\displaystyle\int \frac{1}{(4-x^2)^3}\,dx$

26. $\displaystyle\int \frac{1}{(4x^2+9)^3}\,dx$

27. $\displaystyle\int \sqrt{9+16x^2}\,dx$

28. $\displaystyle\int (9+16x^2)^{3/2}\,dx$

29. $\displaystyle\int \frac{\sqrt{x^2-25}}{x}\,dx$

30. $\displaystyle\int \frac{\sqrt{9x^2-16}}{x}\,dx$

31. $\displaystyle\int x^2\sqrt{x^2-1}\,dx$

32. $\displaystyle\int \frac{x^2}{\sqrt{4x^2-9}}\,dx$

33. $\displaystyle\int \frac{1}{(4x^2-1)^{3/2}}\,dx$

34. $\displaystyle\int \frac{1}{x^2\sqrt{4x^2-9}}\,dx$

35. $\displaystyle\int \frac{\sqrt{x^2-5}}{x^2}\,dx$

36. $\displaystyle\int (4x^2-5)^{3/2}\,dx$

Use hyperbolic substitutions to evaluate the integrals in Problems 37 through 41.

37. $\displaystyle\int \frac{1}{\sqrt{25+x^2}}\,dx$

38. $\displaystyle\int \sqrt{1+x^2}\,dx$

39. $\displaystyle\int \frac{\sqrt{x^2-4}}{x^2}\,dx$

40. $\displaystyle\int \frac{1}{\sqrt{1+9x^2}}\,dx$

41. $\displaystyle\int x^2\sqrt{1+x^2}\,dx$

42. Use the result of Example 2 to show that the area bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

of Fig. 9.6.7 is given by $A = \pi ab$. (The special case $b = a$ is the familiar circular area formula $A = \pi a^2$.)

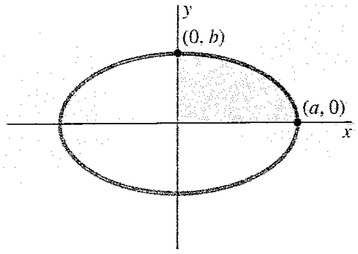


**Fig. 9.6.7** The ellipse of Problem 42

43. Derive the formula $A = \frac{1}{2}a^2\theta$ for the area of a circular sector with radius $a$ and central angle $\theta$ by calculating and adding the areas of the right triangle $OAC$ and the region $ABC$ of Fig. 9.6.8.

44. Compute the arc length of the parabola $y = x^2$ over the interval $[0, 1]$.

45. Compute the area of the surface obtained by revolving around the $x$-axis the parabolic arc of Problem 44.

46. Show that the length of one arch of the sine curve $y = \sin x$ is equal to half the circumference of the ellipse $x^2 + \frac{1}{2}y^2 = 1$. (*Suggestion:* Substitute $x = \cos\theta$ into the arc-length integral for the ellipse.) See Fig. 9.6.9.

47. Compute the arc length of the curve $y = \ln x$ over the interval $[1, 2]$.

$$= \frac{x-1}{3+2x-x^2} + \tfrac{1}{2}\ln\left|\frac{x+1}{\sqrt{3+2x-x^2}}\right| + C. \qquad (2)$$

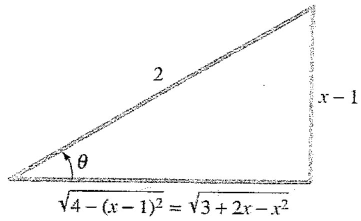


**Fig. 9.7.1** The reference triangle for Example 4.

In the last step we read the values of $\sec\theta$ and $\tan\theta$ from the right triangle in Fig. 9.7.1. When we substitute Eq. (2) into Eq. (1), we finally obtain the result

$$\int \frac{2+6x}{(3+2x-x^2)^2}\,dx = \frac{x+2}{3+2x-x^2} + \frac{1}{2}\ln\left|\frac{x+1}{\sqrt{3+2x-x^2}}\right| + C.$$

The method of Example 4 can be used to evaluate a general integral of the form

$$\int \frac{Ax+B}{(ax^2+bx+c)^n}\,dx, \qquad (3)$$

where $n$ is a positive integer. By splitting such an integral into two simpler ones and then completing the square in the quadratic expression in the denominator, the problem of evaluating the integral in Eq. (3) can be reduced to that of computing

$$\int \frac{1}{(a^2 \pm u^2)^n}\,du. \qquad (4)$$

If the sign in the denominator of Eq. (4) is the plus sign, then the substitution $u = a\tan\theta$ transforms the integral into the form

$$\int \cos^m\theta\,d\theta$$

(see Problem 35). This integral can be handled by the methods of Section 9.4 or by using the reduction formula

$$\int \cos^k\theta\,d\theta = \frac{1}{k}\cos^{k-1}\theta\sin\theta + \frac{k-1}{k}\int \cos^{k-2}\theta\,d\theta$$

of Problem 54 in Section 9.3.

If the sign of the denominator in Eq. (4) is the minus sign, then the substitution $u = a\sin\theta$ transforms the integral into the form

$$\int \sec^m\theta\,d\theta$$

(see Problem 36). This integral may be evaluated with the aid of the reduction formula

$$\int \sec^k\theta\,d\theta = \frac{1}{k-1}\sec^{k-2}\theta\tan\theta + \frac{k-2}{k-1}\int \sec^{k-2}\theta\,d\theta$$

[Eq. (5) of Section 9.3].

## 9.7 PROBLEMS

Evaluate the antiderivatives in Problems 1 through 34.

**1.** $\displaystyle\int \frac{1}{x^2+4x+5}\,dx$

**2.** $\displaystyle\int \frac{2x+5}{x^2+4x+5}\,dx$

**3.** $\displaystyle\int \frac{5-3x}{x^2+4x+5}\,dx$

**4.** $\displaystyle\int \frac{x+1}{(x^2+4x+5)^2}\,dx$

**5.** $\displaystyle\int \frac{1}{\sqrt{3-2x-x^2}}\,dx$

**6.** $\displaystyle\int \frac{x+3}{\sqrt{3-2x-x^2}}\,dx$

**7.** $\displaystyle\int x\sqrt{3-2x-x^2}\,dx$

**8.** $\displaystyle\int \frac{1}{4x^2+4x-3}\,dx$

**9.** $\displaystyle\int \frac{3x+2}{4x^2+4x-3}\,dx$ **10.** $\displaystyle\int \sqrt{4x^2+4x-3}\,dx$

**11.** $\displaystyle\int \frac{1}{x^2+4x+13}\,dx$ **12.** $\displaystyle\int \frac{1}{\sqrt{2x-x^2}}\,dx$

**13.** $\displaystyle\int \frac{1}{3+2x-x^2}\,dx$ **14.** $\displaystyle\int x\sqrt{8+2x-x^2}\,dx$

**15.** $\displaystyle\int \frac{2x-5}{x^2+2x+2}\,dx$ **16.** $\displaystyle\int \frac{2x-1}{4x^2+4x-15}\,dx$

**17.** $\displaystyle\int \frac{x}{\sqrt{5+12x-9x^2}}\,dx$

**18.** $\displaystyle\int (3x-2)\sqrt{9x^2+12x+8}\,dx$

**19.** $\displaystyle\int (7-2x)\sqrt{9+16x-4x^2}\,dx$

**20.** $\displaystyle\int \frac{2x+3}{\sqrt{x^2+2x+5}}\,dx$

**21.** $\displaystyle\int \frac{x+4}{(6x-x^2)^{3/2}}\,dx$ **22.** $\displaystyle\int \frac{x-1}{(x^2+1)^2}\,dx$

**23.** $\displaystyle\int \frac{2x+3}{(4x^2+12x+13)^2}\,dx$ **24.** $\displaystyle\int \frac{x^3}{(1-x^2)^4}\,dx$

**25.** $\displaystyle\int \frac{3x-1}{x^2+x+1}\,dx$ **26.** $\displaystyle\int \frac{3x-1}{(x^2+x+1)^2}\,dx$

**27.** $\displaystyle\int \frac{1}{(x^2-4)^2}\,dx$ **28.** $\displaystyle\int (x-x^2)^{3/2}\,dx$

**29.** $\displaystyle\int \frac{x^2+1}{x^3+x^2+x}\,dx$ **30.** $\displaystyle\int \frac{x^2+2}{(x^2+1)^2}\,dx$

**31.** $\displaystyle\int \frac{2x^2+3}{x^4-2x^2+1}\,dx$ **32.** $\displaystyle\int \frac{x^2+4}{(x^2+1)^2(x^2+2)}\,dx$

**33.** $\displaystyle\int \frac{3x+1}{(x^2+2x+5)^2}\,dx$ **34.** $\displaystyle\int \frac{x^3-2x}{x^2+2x+2}\,dx$

**35.** Show that the substitution $u = a\tan\theta$ gives

$$\int \frac{1}{(a^2+u^2)^n}\,du = \frac{1}{a^{2n-1}}\int \cos^{2n-2}\theta\,d\theta.$$

**36.** Show that the substitution $u = a\sin\theta$ gives

$$\int \frac{1}{(a^2-u^2)^n}\,du = \frac{1}{a^{2n-1}}\int \sec^{2n-1}\theta\,d\theta.$$

In Problems 37 through 39, the region $R$ lies between the curve $y = 1/(x^2-2x+5)$ and the $x$-axis from $x=0$ to $x=5$.

**37.** Find the area of the region $R$.

**38.** Find the volume of the solid generated by revolving $R$ around the $y$-axis.

**39.** Find the volume of the solid generated by revolving $R$ around the $x$-axis.

In Problems 40 through 42 the region $R$ lies between the curve $y = 1/(4x^2-20x+29)$ and the $x$-axis from $x=1$ to $x=4$.

**40.** Find the area of the region $R$.

**41.** Find the volume of the solid generated by revolving $R$ around the $y$-axis.

**42.** Find the volume of the solid generated by revolving $R$ around the $x$-axis.

**43.** Your task is to build a road that joins the points $(0,0)$ and $(3,2)$ and follows the path of the circle with equation $(4x+4)^2+(4y-19)^2 = 377$. Find the length of this road. (Units on the coordinate axes are measured in miles.)

**44.** Suppose that the road of Problem 43 costs $10/(1+x)$ million dollars per mile. (a) Calculate its total cost. (b) With the same cost per mile, calculate the total cost of a straight line road from $(0,0)$ to $(3,2)$. You should find that it is *more* expensive than the *longer* circular road!

In Problems 45 through 47, factor the denominator by first noting by inspection a root $r$ of the denominator and then employing long division by $x-r$. Finally, use the method of partial fractions to aid in finding the indicated antiderivative.

**45.** $\displaystyle\int \frac{3x+2}{x^3+x^2-2}\,dx$

**46.** $\displaystyle\int \frac{1}{x^3+8}\,dx$

**47.** $\displaystyle\int \frac{x^4+2x^2}{x^3-1}\,dx$

**48.** (a) Find constants $a$ and $b$ such that

$$x^4+1 = (x^2+ax+1)(x^2+bx+1).$$

(b) Prove that

$$\int_0^1 \frac{x^2+1}{x^4+1}\,dx = \frac{\pi}{2\sqrt{2}}.$$

(*Suggestion:* If $u$ and $v$ are positive numbers and $uv = 1$, then $\arctan u + \arctan v = \frac{1}{2}\pi$.)

**49.** Factor $x^4+x^2+1$ with the aid of ideas suggested in Problem 48. Then evaluate

$$\int \frac{2x^3+3x}{x^4+x^2+1}\,dx.$$

**50.** Evaluate the integral to show that

$$\int_0^1 \frac{16(x-1)}{x^4-2x^3+4x-4}\,dx = \pi.$$

This integral was (in effect) used by D. Bailey, P. Borwein, and S. Plouffe as a starting point in their recent determination of the 5 billionth hexagesimal digit of the number $\pi$ (it's a 9). (*Suggestion:* Long divide to verify that $x^2-2$ is a factor of the denominator and to find the other factor.)

In Problems 51 through 54, write the general form of a partial fraction decomposition of the given rational function $f(x)$ (with coefficients $A, B, C, \ldots$ remaining to be determined). Then use a

$$W_r = \int_R^r \frac{GMm}{x^2}\,dx.$$

So the work required to move the mass $m$ "infinitely far" from the planet is

$$W = \lim_{r\to\infty} W_r = \int_R^\infty \frac{GMm}{x^2}\,dx = \lim_{r\to\infty}\left[-\frac{GMm}{x}\right]_R^r = \frac{GMm}{R}.$$

Suppose that the mass is projected with initial velocity $v$ straight upward from the planet's surface, as in Jules Verne's novel *From the Earth to the Moon* (1865), in which a spacecraft was fired from an immense cannon. Then the initial kinetic energy $\frac{1}{2}mv^2$ is available to supply this work—by conversion into potential energy. From the equation

$$\frac{1}{2}mv^2 = \frac{GMm}{R},$$

we find that

$$v = \sqrt{\frac{2GM}{R}}.$$

Substitution of appropriate numerical values for the constants $G$, $M$, and $R$ yields the value $v \approx 11175$ mi/s (about 25000 mi/h) for the *escape velocity* from the earth.

## *Present Value of a Perpetuity

Consider a perpetual annuity, under which you and your heirs (and theirs, ad infinitum) will be paid $A$ dollars annually. The question we pose is this: What is the fair market value of such an annuity? What should you pay to purchase it?

If the annual interest rate $r$ is continuously compounded, then a dollar deposited in a savings account would grow to $e^{rt}$ dollars in $t$ years. Hence $e^{-rt}$ dollars deposited now would yield \$1 after $t$ years. Consequently, the **present value** of the amount you (and your heirs) will receive between time $t = 0$ (the present) and time $t = T > 0$ is defined to be

$$P_T = \int_0^T Ae^{-rt}\,dt.$$

Hence the total present value of the perpetual annuity is

$$P = \lim_{T\to\infty} P_T = \int_0^\infty Ae^{-rt}\,dt = \lim_{T\to\infty}\left[-\frac{A}{r}e^{-rt}\right]_0^T = \frac{A}{r}.$$

Thus $A = rP$. For instance, at an annual interest rate of 8% ($r = 0.08$), you should be able to purchase for $P = (\$50000)/(0.08) = \$625{,}000$ a perpetuity that pays you (and your heirs) an annual sum of \$50000 forever.

## 9.8 PROBLEMS

Determine whether the improper integrals in Problems 1 through 38 converge. Evaluate those that do converge.

**1.** $\displaystyle\int_2^\infty \frac{1}{x\sqrt{x}}\,dx$

**2.** $\displaystyle\int_1^\infty \frac{1}{x^{2/3}}\,dx$

**3.** $\displaystyle\int_0^4 \frac{1}{x\sqrt{x}}\,dx$

**4.** $\displaystyle\int_0^8 \frac{1}{x^{2/3}}\,dx$

**5.** $\displaystyle\int_1^\infty \frac{1}{x+1}\,dx$

**6.** $\displaystyle\int_3^\infty \frac{1}{\sqrt{x+1}}\,dx$

**7.** $\displaystyle\int_5^\infty \frac{1}{(x-1)^{3/2}}\,dx$

**8.** $\displaystyle\int_0^4 \frac{1}{\sqrt{4-x}}\,dx$

9. $\int_0^9 \frac{1}{(9-x)^{3/2}}\,dx$

10. $\int_0^3 \frac{1}{(x-3)^2}\,dx$

11. $\int_{-\infty}^{-2} \frac{1}{(x+1)^3}\,dx$

12. $\int_{-\infty}^{0} \frac{1}{\sqrt{4-x}}\,dx$

13. $\int_{-1}^{8} \frac{1}{\sqrt[3]{x}}\,dx$

14. $\int_{-4}^{4} \frac{1}{(x+4)^{2/3}}\,dx$

15. $\int_2^\infty \frac{1}{\sqrt[3]{x-1}}\,dx$

16. $\int_{-\infty}^{\infty} \frac{x}{(x^2+4)^{3/2}}\,dx$

17. $\int_{-\infty}^{\infty} \frac{x}{x^2+4}\,dx$

18. $\int_0^\infty e^{-(x+1)}\,dx$

19. $\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

20. $\int_0^2 \frac{x}{x^2-1}\,dx$

21. $\int_0^\infty xe^{-3x}\,dx$

22. $\int_{-\infty}^{2} e^{2x}\,dx$

23. $\int_0^\infty xe^{-x^2}\,dx$

24. $\int_{-\infty}^{\infty} |x|e^{-x^2}\,dx$

25. $\int_0^\infty \frac{1}{1+x^2}\,dx$

26. $\int_0^\infty \frac{x}{1+x^2}\,dx$

27. $\int_0^\infty \cos x\,dx$

28. $\int_0^\infty \sin^2 x\,dx$

29. $\int_1^\infty \frac{\ln x}{x}\,dx$

30. $\int_2^\infty \frac{1}{x\ln x}\,dx$

31. $\int_2^\infty \frac{1}{x(\ln x)^2}\,dx$

32. $\int_1^\infty \frac{\ln x}{x^2}\,dx$

33. $\int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\,dx$

34. $\int_0^{\pi/2} \frac{\sin x}{(\cos x)^{4/3}}\,dx$

35. $\int_0^1 \ln x\,dx$

36. $\int_0^1 \frac{\ln x}{x}\,dx$

37. $\int_0^1 \frac{\ln x}{x^2}\,dx$

38. $\int_0^\infty e^{-x}\cos x\,dx$

In Problems 39 through 42, the given integral is improper both because the interval of integration is unbounded and because the integrand is unbounded near zero. Investigate its convergence by expressing it as a sum of two integrals—one from 0 to 1, the other from 1 to ∞. Evaluate those integrals that converge.

39. $\int_0^\infty \frac{1}{x+x^2}\,dx$

40. $\int_0^\infty \frac{1}{x^2+x^4}\,dx$

41. $\int_0^\infty \frac{1}{x^{1/2}+x^{3/2}}\,dx$

42. $\int_0^\infty \frac{1}{x^{2/3}+x^{4/3}}\,dx$

In Problems 43 through 46, find all real number values of $k$ for which the given improper integral converges. Evaluate the integral for those values of $k$.

43. $\int_0^1 \frac{1}{x^k}\,dx$

44. $\int_1^\infty \frac{1}{x^k}\,dx$

45. $\int_0^1 x^k \ln x\,dx$

46. $\int_1^\infty \frac{1}{x(\ln x)^k}\,dx$

47. Beginning with the definition of the gamma function in Eq. (7), integrate by parts to show that

$$\Gamma(x+1) = x\Gamma(x)$$

for every positive real number $x$.

48. Explain how to apply the result of Problem 47 $n$ times in succession to show that if $n$ is a positive integer, then $\Gamma(n+1) = n!\Gamma(1) = n!$.

Problems 49 through 51 deal with Gabriel's horn, the surface obtained by revolving the curve $y = 1/x$, $x \geqq 1$, around the $x$-axis (Fig. 9.8.12).

49. Show that the area under the curve $y = 1/x$, $x \geqq 1$, is infinite.

50. Show that the volume of revolution enclosed by Gabriel's horn is finite, and compute it.

51. Show that the surface area of Gabriel's horn is infinite. (*Suggestion:* Let $A_t$ denote the surface area from $x = 1$ to $x = t > 1$. Prove that $A_t > 2\pi \ln t$.) In any case, the implication is that we could fill Gabriel's horn with a finite amount of paint (Problem 50), but no finite amount suffices to paint its surface.

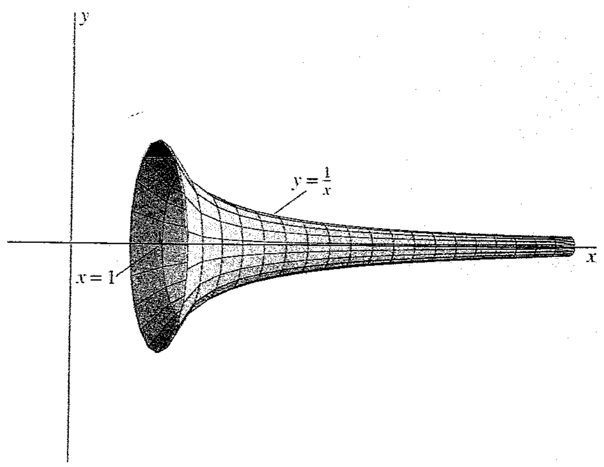


**Fig. 9.8.12** Gabriel's horn (Problems 49 through 51)

52. Show that

$$\int_{-\infty}^{\infty} \frac{1+x}{1+x^2}\,dx$$

diverges, but that

$$\lim_{t\to\infty} \int_{-t}^{t} \frac{1+x}{1+x^2}\,dx = \pi.$$

53. Use the substitution $x = e^{-u}$ and the fact that $\Gamma(n+1) = n!$ (Problem 48) to prove that if $m$ and $n$ are fixed but arbitrary positive integers, then