1. Compute

(a) $\lim_{n\to\infty} 1 + \frac{1}{n} + \frac{1}{n^2}$

(b) $\lim_{a\to -1} \frac{a^2+4a+3}{a^2-2a-3}$

(c) $\lim_{\theta\to 0} \frac{\theta - \sin\theta}{\theta^3}$

(5+5+5)

(a) $\lim_{n\to\infty} \frac{1}{n} = 0$ and $\lim_{n\to\infty} \frac{1}{n^2} = 0$, so $\lim_{n\to\infty} 1+\frac{1}{n}+\frac{1}{n^2} = 1$

(b) $\lim_{a\to -1} \frac{a^2+4a+3}{a^2-2a-3} = \lim_{a\to -1} \frac{(a+1)(a+3)}{(a+1)(a-3)} = \lim_{a\to -1} \frac{a+3}{a-3} = \frac{-1+3}{-1-3}$

$= -\frac{2}{4} = -\frac{1}{2}$

(c) Solution 1: Use Power series $\sin\theta = \theta - \frac{1}{3!}\theta^3 + O(\theta^5)$,

so $\frac{\theta - \sin\theta}{\theta^3} = \frac{1}{3!} + O(\theta^2) \xrightarrow[n\to\infty]{} \frac{1}{6}$

Solution 2: Use L'Hôpital's rule:

$f(\theta) = \theta - \sin\theta \Rightarrow f'(\theta) = 1 - \cos\theta \Rightarrow f''(\theta) = \sin\theta$
$\Rightarrow f'''(\theta) = \cos\theta \Rightarrow f'''(0) = 1$

$g(\theta) = \theta^3 \Rightarrow g'(\theta) = 3\theta^2\ 2 \Rightarrow g''(\theta) = 6\theta \Rightarrow g'''(\theta) = 6$

Then $\lim_{\theta\to 0} \frac{\theta - \sin\theta}{\theta^3} = \frac{f'''(0)}{g'''(0)} = \frac{1}{6}$

2. Consider the function

$$f(x) = \begin{cases} 0 & \text{for } x \le 0 \\ e^{-1/x} & \text{for } x > 0\,. \end{cases}$$

(a) Is f continuous at $x = 0$? Explain.

(b) Is f differentiable at $x = 0$? If so, find $f'(0)$.

(5+5)

(a) $\lim\limits_{x \to 0^-} f(x) = 0$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} e^{-\frac{1}{x}} = \lim\limits_{y \to \infty} e^{-y} = 0$

So $\lim\limits_{x \to 0} f(x) = 0 = f(0)$; f is continuous.

(b) The left-sided derivative is obviously 0.

So let's consider the right-sided derivative

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{e^{-\frac{1}{h}}}{h}$$

$$= \lim_{z \to \infty} z e^{-z} = 0$$

So the derivative at $x=0$ exists and $f'(0) = 0$.

3. Consider the graphs of the functions

$$f(x) = mx$$

and

$$g(x) = \arctan x.$$

For which values of m do they intersect in (a) one, (b) two, and (c) three points? (10)

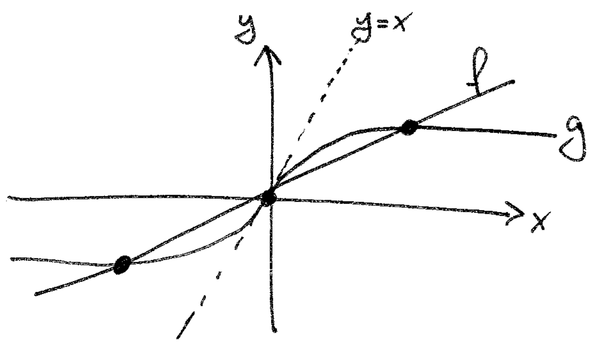
$g'(x) = \frac{1}{1+x^2}$ (see front page ...)

$\Rightarrow g'(x) \leq 1$ with $g'(0) = 1$

$g''(x) = \frac{-2x}{(1+x^2)^2}$, so g is concave down in right half-plane
and g " " up " left " "

So if $m \leq 0$ or $m \geq 1$, there is only the trivial intersection point at $x=0$.

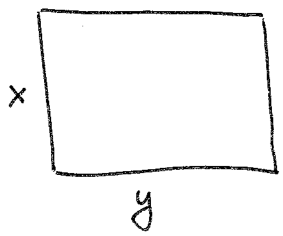
If $m \in (0,1)$, we have qualitatively the following situation:



$m = f'(0) < g'(0) = 1$

so there must be at least two more intersection points as arctan has a horizontal asymptote as $x \to \pm\infty$. There are no more than those three intersection points as g does not change concavity in each half-plane.

4. Show that the square is the rectangle of maximal area for a given circumference. (10)



$$2x + 2y = C$$

$$xy = A$$

$$\Rightarrow \quad A(x) = x\left(\frac{C}{2} - x\right)$$

$$\Rightarrow \quad A'(x) = \frac{C}{2} - 2x$$

$A'(x) = 0$ implies $\frac{C}{2} = 2x \Rightarrow x = \frac{C}{4}$

This is a square!

Clearly, $A(x)$ is a parabola open to the bottom, so there must be a unique maximum of the area.

5. For each $a > 0$, consider the function

$$f(p) = p\,a^{1/p} = p\, e^{\ln a / p}$$

Find all minima and maxima, points of inflection, and vertical and horizontal asymptotes on the interval $p \in (0, \infty)$. (10)

$$f'(p) = a^{\frac{1}{p}} + p \,\frac{-\ln a}{p^2}\, e^{\frac{\ln a}{p}} = a^{\frac{1}{p}}\left(1 - \frac{\ln a}{p}\right)$$

$$f''(p) = \frac{-\ln a}{p^2}\, a^{\frac{1}{p}}\left(1 - \frac{\ln a}{p}\right) + a^{\frac{1}{p}}\left(\frac{\ln a}{p^2}\right)$$

$$= \frac{\ln a}{p^2}\left[-\left(1 - \frac{\ln a}{p}\right) + 1\right] a^{\frac{1}{p}}$$

$$= -\frac{(\ln a)^2}{p^3}\, a^{\frac{1}{p}} \neq 0 \quad \Rightarrow \text{ no points of inflection.}$$

Case $a \leqslant 1$: $f'(p) > 0$ and $\lim_{p \to 0} f(p) = 0$ with $\lim_{p \to \infty} f(p) = \infty$

$\Rightarrow$ no extreme points

Case $a > 1$: $f'(p) = 0$ when $p = \ln a$.

Since $\lim_{p \to 0} f(p) = \exp\left(\lim_{p \to 0} \ln f(p)\right)$

$$= \exp\left(\lim_{p \to 0}\left(\ln p + \frac{\ln a}{p}\right)\right) = \infty,$$

we have a vertical asymptote at $p = 0$ and a minimum at $(\ln a, f(\ln a)) = (\ln a, e \ln a)$.

No maxima or horizontal asymptotes.

6. Find the indefinite integral

$$\int \cos^2\theta \, d\theta$$

by

(a) using trigonometric identities, and

(b) writing the expression in terms of complex exponentials. You should

(c) verify that the results from (a) and (b) are compatible.

(5+5+5)

(a) $\int \cos^2\theta \, d\theta = \frac{1}{2}\int d\theta + \frac{1}{2}\int \cos 2\theta \, d\theta$

$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C$

(b) $\int \cos^2\theta \, d\theta = \int \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^2 d\theta$

$= \frac{1}{4}\int \left(e^{2i\theta} + 2 + e^{-2i\theta}\right) d\theta$

$= \frac{1}{4}\left[\frac{e^{2i\theta}}{2i} + 2\theta + \frac{e^{-2i\theta}}{-2i}\right] + C$

$= \frac{\theta}{2} + \frac{1}{4} \underbrace{\frac{e^{2i\theta} - e^{-2i\theta}}{2i}}_{= \sin 2\theta} + C$

so the solutions coincide.

7. Find the indefinite integrals

(a) $\int \frac{1}{1+e^x} \, dx$

(b) $\int \frac{4x-2}{x^3-x} \, dx$

(5+5)

(a) Substitute $u = e^x \Rightarrow du = e^x \, dx \Rightarrow du = u \, dx$

$$\Rightarrow \int \frac{1}{1+e^x} dx = \int \frac{1}{u(1+u)} du$$

$$= \int \frac{1}{u} du + \int \frac{-1}{1+u} du$$

$$= \ln u - \ln(1+u) + C$$

$$= x - \ln(1+e^x) + C$$

(b) Denominator factorizes as $x^3 - x = x(x+1)(x-1)$

Try partial fractions

$$\frac{4x-2}{x^3-x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} \Rightarrow A(x^2-1) + B(x-1)x + Cx(x+1) = 4x-$$

$(A+B+C)x^2 = 0$

$(-B+C)x = 4x$

$-A = -2 \Rightarrow A = 2$

$\Rightarrow B+C = -2 \quad C-B = 4 \Rightarrow 2C = 2 \Rightarrow C = 1 \Rightarrow B = -3$

So:

$$\int \frac{4x-2}{x^3-x} dx = 2\int \frac{1}{x} dx - 3\int \frac{1}{x+1} dx + \int \frac{1}{x-1} dx$$

$$= 2\ln x - 3\ln(x+1) + \ln(x-1) + C$$

8. Compute the Taylor series of

   (a) $f(x) = (x+2)(x-2)$,
   (b) $g(x) = \ln(1+x)$,
   (c) $h(x) = \ln(x+1) + \ln(1/(x+1))$.

$\Bigg\}$ about $x=0$

(5+5+5)

9. (a) Does the infinite series

$$\sum_{n=0}^{\infty} \frac{n}{n^2+1}$$

   converge?

   (b) Show that

$$\sum_{n=0}^{\infty} \left(\frac{\sin n}{2}\right)^n \leq 2$$

(5+5)

8(a) $f(x) = x^2 - 4$ is a polynomial, so it is its own Taylor series

(b) $g'(x) = \frac{1}{1+x}$ $\quad g''(x) = -\frac{1}{(1+x)^2}$ $\quad g'''(x) = 2\frac{1}{(1+x)^3}$ $\quad \ldots$

$\Rightarrow \ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$

(c) $\ln(x+1) + \ln\left(\frac{1}{x+1}\right) = \ln\left(\frac{x+1}{x+1}\right) = 0$, so the Taylor series is 0 as well.

9 (a) $\frac{n}{n^2+1} > \frac{1}{2n}$ for $n=0$ and the harmonic series $\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent, so $\sum_{n=0}^{\infty}\frac{n}{n^2+1}$ diverges as well.

(b) $\left|\frac{1}{2}\sin n\right|^n \leqslant \frac{1}{2}^n$

geometric series!

$\Rightarrow \sum_{n=0}^{\infty}\left(\frac{\sin n}{2}\right)^n \leqslant \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$

10. *Note:* In parts (a) and (b) of the following problem you may leave logarithms and exponential functions unevaluated.

The concentration $c(t)$ of a chemical catalyst decays at a rate $\lambda\, c(t)$, where $\lambda = 1\,\mathrm{s}^{-1}$.

(a) If the initial concentration of the catalyst is $c(0) = 10\,\mathrm{g/l}$, what is the concentration after 2 s?

(b) How long does it take until half of the catalyst has decayed?

(c) Suppose the catalyst is added to a reactant with concentration $r(t)$ which undergoes a chemical reaction at rate $\nu\, c(t)\, r(t)$, where ν is some given constant. Show that the fraction of the reactant which remains after all chemical activity has subsided is given by the expression $e^{-\nu c(0)/\lambda}$.

(5+5+5)

(a) $$\frac{dc(t)}{dt} = -\lambda\, c(t) \quad \Rightarrow \quad c(t) = c(0)\, e^{-\lambda t}$$

Here: $c(2) = 10\, e^{-2}\ \frac{g}{\ell}$

(b) $$c(t_{1/2}) = \frac{1}{2} c(0) \quad \Rightarrow \quad e^{-\lambda t_{1/2}} = \frac{1}{2} \quad \Rightarrow \quad \lambda t_{1/2} = \ln 2$$

$$\Rightarrow t_{1/2} = \frac{\ln 2}{\lambda}$$ Here: $t_{1/2} = \ln 2\ s$

(c) $$\frac{dr}{dt} = -\nu\, c(t)\, r(t) \quad \Rightarrow \quad \frac{dr}{r} = -\nu\, c(0)\, e^{-\lambda t}\, dt$$

$$\Rightarrow \int_{r(0)}^{r(\infty)} \frac{dr}{r} = -\nu\, c(0) \int_0^\infty e^{-\lambda t}\, dt$$

$$\Rightarrow \ln r \Big|_{r(0)}^{r(\infty)} = \frac{\nu c(0)}{\lambda}\, e^{-\lambda t} \Big|_0^\infty$$

$$\Rightarrow \ln \frac{r(\infty)}{r(0)} = -\frac{\nu\, c(0)}{\lambda} \quad \Rightarrow \quad \frac{r(\infty)}{r(0)} = e^{-\frac{\nu c(0)}{\lambda}}$$