1. Let X be a Banach space such that for some $U \subset \mathbb{R}^n$ open and $1 \leq p < p \leq \infty$, X is compactly embedded in $L^p(U)$ and X is continuously embedded in $L^q(U)$.

   Prove that X is compactly embedded in any $L^r(U)$ with $r \in [p, q)$. (5)

We need to show that $u_n \rightharpoonup 0$ weakly in X implies that $u_n \to 0$ strongly in L.

By the L^p-interpolation inequality,

$$\|u_n\|_{L^r} \leq \underbrace{\|u_n\|_{L^p}^{\theta}}_{\downarrow\, 0} \underbrace{\|u_n\|_{L^q}^{1-\theta}}_{\leq C} \quad \text{where } \frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$$

$\leq C$ as $u_n \overset{X}{\rightharpoonup} 0$ implies u_n is L^q-bounded.

0 as $u_n \overset{X}{\rightharpoonup} 0$ implies, by assumption, $u_n \xrightarrow{L^p} 0$

So $u_n \to 0$ in L^r.

2. Let

$$U = \{x \in \mathbb{R}^n : 0 < x_n < L\}$$

denote the region between two parallel hyperplanes. Show that, for all $u \in \cancel{H^1_0(U)}$, $W_0^{1,p}(\bar{U})$

$$\|u\|_{L^p(U)} \leq \frac{L}{p^{1/p}} \|Du\|_{L^p(U)}.$$

(5)

Assume first that $u \in C_c^\infty(\bar{U})$. Then

$$u(x) = \int_0^{x_n} \partial_{x_n} u(x_1, \dots, x_{n-1}, t)\, dt$$

$$\Rightarrow \int_U |u(x)|^p\, dx \leq \int_{\mathbb{R}^{n-1}} \int_0^L \underbrace{\left(\int_0^{x_n} |\partial_n u(x_1, \dots, x_{n-1}, t)|\, dt \right)^p}\, dx_n\, dx'$$

$$\leq \left(\int_0^{x_n} dt \right)^{1-\frac{1}{p}} \left(\int_0^{x_n} |\partial_n u|^p \right)^{\frac{1}{p}}$$

$$\leq \int_{\mathbb{R}^{n-1}} \int_0^L x_n^{p-1}\, dx_n \int_0^L |\partial_n u|^p\, dx' \qquad x' = (x_1, \dots, x_{n-1})$$

$$\leq \frac{L^p}{p} \|Du\|_{L^p}^p$$

Now take the p-th root. Since $C_c^\infty(\bar{U})$ is dense $W_0^{1,p}(\bar{U})$, the estimate extends to $u \in W_0^{1,p}(\bar{U})$.

3. Let $U \subset \mathbb{R}^n$ be open and bounded. Show that λ is the smallest eigenvalue of the Dirichlet Laplacian, i.e., is the smallest real number such that

$$-\Delta u - \lambda u = 0 \quad \text{in } U, \qquad u = 0 \quad \text{on } \partial U$$

has weak solutions $u \in H_0^1(U)$ if and only if $1/\lambda$ is the smallest constant c such that

$$\|u\|^2_{L^2(U)} \le c\,\|Du\|^2_{L^2(U)} \qquad (**)$$

for all $u \in H_0^1(U)$. (5)

We know from class that the Dirichlet Laplacian has a complete L^2-orthonormal set of eigenfunctions $e_i \in H_0^1(U)$ with corresponding eigenvalues $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \dots$, so that

$$\langle Le_i, v\rangle_{H^{-1},H_0^1} \equiv \int_U De_i \cdot Dv\,dx - \lambda_i \int_U e_i v\,dx = 0 \quad \forall v \in H_0^1(U)$$

Then, for any $v \in H_0^1(U)$, we can write $v = \sum_{i=1}^\infty v_i e_i$, so that

$$\|Dv\|^2_{L^2} = \sum_{i=1}^\infty v_i \int_U De_i \cdot Dv\,dx = \sum_{i=1}^\infty \lambda_i v_i \int_U e_i v\,dx$$

$$= \sum_{i=1}^\infty \lambda_i |v_i|^2 \ge \lambda_1 \sum_{i=1}^\infty |v_i|^2 = \lambda_1 \|v\|^2_{L^2} \qquad (*)$$

This inequality saturates when $v = v_1$, so λ_1 is the best constant.

Vice versa, it is immediate from $(*)$ that if λ is not the smallest eigenvalue, then $c = \frac{1}{\lambda}$ cannot be best constant for $(**)$.

4. Let $U \subset \mathbb{R}^n$ be open and bounded with C^1 boundary. We say that $u \in H^1(U)$ is a weak solution if

$$\int_U Du \cdot Dv \, dx + \int_U uv \, dx + \int_{\partial U} uv \, dS = \int_U fv \, dx \qquad (*)$$

for all $v \in H^1(U)$.

(a) Assume, moreover, that $u \in C^2(U) \cap C^1(\bar{U})$. Which second order boundary value problem corresponds to the weak formulation above? (5)

After integration by parts, $(*)$ reads

$$\int_U (-\Delta u + u - f) \, v \, dx + \int_{\partial U} (\nu \cdot Du + u) v \, dx = 0 .$$

First, take test functions $v \in H_0^1(U)$, where the boundary integral vanishes.

$$\Rightarrow \quad -\Delta u + u = f \quad \text{in } U$$

Then the boundary integral must vanish independently, so that

$$\nu \cdot Du + u = 0 \quad \text{on } \partial U$$

(b) Prove that the weak formulation has a unique solution u for every $f \in L^2(U)$. (5)

Since

$$\int_{\partial U} uv\, dS \le \|u\|_{L^2(\partial U)} \|v\|_{L^2(\partial U)}$$

$$\le c\, \|u\|_{H^1(U)} \|v\|_{H^1(U)} \quad \text{by the trace theorem,}$$

the LHS of (*) is a continuous bilinear form on $H^1(U)$.

It is also coercive, as

$$\int_U Du\, Du\, dx + \int_U u^2\, dx + \int_{\partial U} u^2\, dS \ge \|u\|_{H^1}^2 .$$

The claim thus follows by the Lax-Milgram theorem.

5. Let

$$U = \{x \in \mathbb{R}^3 : |x| \le \pi\}.$$

Show that the problem

$$-\Delta u - u = f \qquad \text{in } U,$$
$$u = 0 \qquad \text{on } \partial U$$

has a weak solution in $H_0^1(U)$ only if

$$\int_U f(x) \underbrace{\frac{\sin(|x|)}{|x|}}_{=: v} dx = 0. \tag{5}$$

Note that $L = -\Delta - 1$ is formally self-adjoint, and that

$$L^* v = -\Delta v - v = -D\cdot\left(\left(\frac{\cos|x|}{|x|} - \frac{\sin|x|}{|x|^2}\right)\frac{x}{|x|}\right) - v$$

$$= -\left[\underset{\substack{\| \\ 3}}{n}\left(\frac{\cos|x|}{|x|^2} - \frac{\sin|x|}{|x|^3}\right) + \left(\frac{-\sin|x|}{|x|^2} - 2\cos|x|\,|x|^{-3}\right)\underbrace{\frac{x}{|x|}\cdot x}_{=|x|}\right.$$
$$\left. - \left(\frac{\cos|x|}{|x|^3} - 3\frac{\sin|x|}{|x|^4}\right)\underbrace{\frac{x}{|x|}\cdot x}_{=|x|}\right] - v$$

$$= \frac{\sin|x|}{|x|} - v = 0$$

Thus, $v \in \operatorname{Ker} L^*$, so $f \perp v$ is a necessary condition for solvability.