Partial Differential Equations

Midterm Exam

October 27, 2009

1. Let X be a Banach space such that for some $U \subset \mathbb{R}^n$ open and $1 \leq p < p \leq \infty$, X is compactly embedded in $L^p(U)$ and X is continuously embedded in $L^q(U)$.

   Prove that X is compactly embedded in any $L^r(U)$ with $r \in [p, q)$. (5)

2. Let
$$U = \{x \in \mathbb{R}^n : 0 < x_n < L\}$$
denote the region between two parallel hyperplanes. Show that, for all $u \in W_0^{1,p}(U)$,
$$\|u\|_{L^p(U)} \leq \frac{L}{p^{1/p}} \|Du\|_{L^p(U)} .$$
(5)

3. Let $U \subset \mathbb{R}^n$ be open and bounded. Show that λ is the smallest eigenvalue of the Dirichlet Laplacian, i.e., is the smallest real number such that
$$-\Delta u - \lambda u = 0 \qquad \text{in } U\,,$$
$$u = 0 \qquad \text{on } \partial U$$
has weak solutions $u \in H_0^1(U)$ if and only if $1/\lambda$ is the smallest constant c such that
$$\|u\|_{L^2(U)}^2 \leq c \, \|Du\|_{L^2(U)}^2$$
for all $u \in H_0^1(U)$. (5)

4. Let $U \subset \mathbb{R}^n$ be open and bounded with C^1 boundary. We say that $u \in H^1(U)$ is a weak solution if
$$\int_U Du \cdot Dv \, dx + \int_U u \, v \, dx + \int_{\partial U} u \, v \, dS = \int_U f \, v \, dx$$
for all $v \in H^1(U)$.

   (a) Assume, moreover, that $u \in C^2(U) \cap C^1(\bar{U})$. Which second order boundary value problem corresponds to the weak formulation above?

(b) Prove that the weak formulation has a unique solution u for every $f \in L^2(U)$. (5)

5. Let
$$U = \{x \in \mathbb{R}^3 \colon |x| \leq \pi\} \,.$$
Show that the problem
$$\begin{aligned} -\Delta u - u &= f \qquad \text{in } U \,, \\ u &= 0 \qquad \text{on } \partial U \end{aligned}$$
has a weak solution in $H_0^1(U)$ only if
$$\int_U f(x) \, \frac{\sin(|x|)}{|x|} \, dx = 0 \,.$$
(5)