1. A "totally unlucky" number is one that contains no seven in any decimal expansion. Compute the Lebesgue measure of the totally unlucky numbers in $[0,1]$. (10)

Let $\sigma = \frac{1}{10}$.

Then the Lebesgue measure of the not totally unlucky numbers is

$$
\begin{aligned}
L = {} & \sigma && \text{(first digit after decimal point is 7)} \\
& + (1-\sigma)\sigma && \text{(second digit after decimal point is 7,} \\
& + (1-\sigma)(1-\sigma)\sigma + \dots \\
= {} & \sigma \sum_{i=1}^{\infty} (1-\sigma)^i = \sigma \frac{1}{1-(1-\sigma)} = 1
\end{aligned}
$$

So the totally unlucky numbers have Lebesgue measure 0.

<u>Alternative Solution:</u> Let E be the set of totally unlucky numbers. Assume $\mu(E)>0$. Then, by inner regularity of the Lebesgue measure, there exists $K \subset E$, K compact, with $\mu(K)>0$. Since every compact set of the real line is a countable disjoint union of closed intervals or points (this is e.g. the complement of Folland, Proposition 0.24), there must be $[a,b] \subset K$, $a<b$. But intervals always contain numbers which are not totally unlucky. Contradiction. $\square$

2. Let (Ω, Σ, μ) be a measure space, $1 \le p < \infty$, and $f\colon \Omega \to [0,\infty]$ a measurable function. Show that

$$\mu\{x \in \Omega \colon f(x) > \alpha\} \le \frac{1}{\alpha^p} \int_\Omega f^p \, d\mu.$$

(10)

$$
\begin{aligned}
\int_\Omega f^p \, d\mu &= \int_0^\infty p\, t^{p-1} \mu\{x \in \Omega : f(x) > t\} \, dt \\
&\geq \int_0^\alpha p\, t^{p-1} \mu\{x \in \Omega : f(x) > t\} \, dt \\
&\geq \mu\{x \in \Omega : f(x) > \alpha\} \; \underbrace{p \int_0^\alpha t^{p-1} \, dt}_{= \alpha^p}
\end{aligned}
$$

$\square$

3. Let (Ω, Σ, μ) be a measure space and $f\colon \Omega \to [0,\infty]$ a measurable function with

$$0 < \int_\Omega f \, d\mu < \infty .$$

Find, with justification,

$$\underbrace{\lim_{n\to\infty} \int_\Omega n \ln\left(1 + \left(\frac{f}{n}\right)^\alpha\right) d\mu}_{=: I_n}$$

when

(a) $\alpha = 1$,

(b) $0 < \alpha < 1$,

(c) $\alpha > 1$.

(5+5+5)

(a) $$n \ln\left(1 + \frac{f}{n}\right) = \ln\left(\left(1+\frac{f}{n}\right)^n\right) \nearrow \ln e^f = f$$

since $\left(1+\frac{x}{n}\right)^n \nearrow e^x$

$\Rightarrow \quad I_n \to \int_\Omega f \, d\mu$ by monotone convergence

(b) The pointwise limit is ∞ whenever $f(x) > 0$. Since this is ∞ on a set of positive measure, we apply Fatou's Lemma to conclude

$$\lim_{n\to\infty} I_n \geqslant \int_{\{x\in\Omega:\, f(x)>0\}} \lim_{n\to\infty} n \ln\left(1+\left(\frac{f}{n}\right)^\alpha\right) d\mu = \infty .$$

(c) The pointwise limit is 0 (L'Hôpital's rule). We want to conclude that $I_n \to 0$ by dominated convergence.

<u>Claim</u>: αf is a dominating function.

<u>Proof</u>: $1 + x^\alpha \leq (1+x)^\alpha \Rightarrow \ln(1+x^\alpha) \leq \alpha \ln(1+x) \leq \alpha x$. Now set $x = \frac{f}{n}$.

<u>Long Proof of Claim</u>: We first show that

$$g(x) = \alpha x - \ln(1 + x^\alpha)$$

is non-negative for $x \geqslant 0$. This is so because $g(0) = 0$

and
$$g'(x) = \alpha - \frac{\alpha x^{\alpha-1}}{1+x^\alpha}$$

$$= \frac{\alpha}{1+x^\alpha}\left[1 + x^\alpha - x^{\alpha-1}\right]$$

$$= \frac{\alpha}{1+x^\alpha}\underbrace{\left[1 + x^{\alpha-1}(x-1)\right]}_{\geqslant 0} \geqslant 0 .$$

Now set $x = \frac{f}{n}$:

$$\Rightarrow \quad \alpha \frac{f}{n} - \ln\left(1 + \left(\frac{f}{n}\right)^\alpha\right) \geqslant 0$$

$$\Rightarrow \quad \alpha f \geqslant n \ln\left(1 + \left(\frac{f}{n}\right)^\alpha\right) \geqslant 0$$

$\square$

4. Let $1 < p < \infty$. For $f \in L^p((0,\infty))$, which for simplicity you may assume nonnegative, define the local average function

$$F(x) = \frac{1}{x}\int_0^x f(\xi)\,d\xi\,.$$

Show that $F \in L^p((0,\infty))$ with

$$\|F\|_p \le \frac{p}{p-1}\|f\|_p\,.$$

Hint: Find a differential equation for F, then multiply by F^{p-1} and integrate. You may assume first that $f \in C_c^\infty((0,\infty))$, then extend by density. (10)

$$x\,F(x) = \int_0^x f(\xi)\,d\xi \quad \Rightarrow \quad F + xF' = f$$

$$\Rightarrow \underbrace{\int_0^\infty F\,F^{p-1}\,dx}_{=\|F\|_p^p} + \underbrace{\int_0^\infty x\,F'\,F^{p-1}\,dx}_{=\frac{1}{p}\int_0^\infty x\frac{d}{dx}F^p\,dx} = \int_0^\infty f\,F^{p-1}\,dx$$

$$= -\frac{1}{p}\int_0^\infty F^p\,dx \quad \text{if } f \text{ is compactly supported so that } F(x)\to 0 \text{ as } x\to\infty$$

$$\Rightarrow \underbrace{\left(1-\frac{1}{p}\right)}_{=\frac{p-1}{p}}\|F\|_p^p = \int_0^\infty f\,F^{p-1}\,dx \le \|f\|_p\,\underbrace{\|F^{p-1}\|_{p'}}_{=\|F\|_p^{p-1}}\ ; \ \frac{1}{p}+\frac{1}{p'}=1$$

$$\Rightarrow \|F\|_p \le \frac{p}{p-1}\|f\|_p$$

Since C_c^∞ is dense in L^p, the claim extends to any $f \in L^p$.

5. Let H be a Hilbert space and $V \subset H$ a closed subspace. In linear algebra, the quotient space H/V is defined as the space of cosets $\{x+V : v \in V\}$. (In other words, it is the set of equivalence classes with respect to the equivalence relation $x \sim y$ if $x - y \in V$.)

How can you define an inner product on H/V so that the *quotient map* $p: H \to H/V$ defined by $p(x) = x + V$ is continuous? (10)

We know that any $x \in H$ can be uniquely decomposed as

$$x = x_\parallel + x_\perp \quad , \quad \text{where } x_\parallel \in V,\ x_\perp \in V^\perp .$$

Thus, define

$$\langle x+V, y+V\rangle_{H/V} \equiv \langle x_\perp, y_\perp\rangle_H$$

Then

$$\|p(x)\|_{H/V}^2 = \|x_\perp\|_H^2 \le \|x\|_H^2 ,$$

hence p is a bounded, thus continuous, linear map.

Note that $\langle x+V, x+V\rangle_{H/V} = 0$ implies $x_\perp = 0$

$$\Rightarrow x \in 0 + V , \text{ the zero element in } H/V.$$

The other properties of the inner product follow readily from the definition.

6. Let H be a Hilbert space, $\{x_n\}_{n\in\mathbb{N}} \subset H$ and $x \in H$ such that

$$x_n \rightharpoonup x$$

weakly.

(a) Show that

$$\|x\| \le \liminf_{n\to\infty} \|x_n\| .$$

(b) Give an example where (a) holds with strict inequality.

(10+5)

(a) $\|x\|^2 = \langle x, x\rangle$

$= \lim_{n\to\infty} \langle x, x_n\rangle$ as $x_n \rightharpoonup x$

$\le \liminf_{n\to\infty} \|x\|\,\|x_n\|$ by Cauchy-Schwarz.

$\Rightarrow \|x\| \le \liminf_{n\to\infty} \|x_n\|$ $\square$

(b) $H = L^2(\mathbb{T})$, $\cancel{\&}$ $f_n(x) = \cos nx$

$\Rightarrow f_n \rightharpoonup 0$ weakly (shown in class)

but $\displaystyle\int_0^{2\pi} \cos^2 nx \, dx = \int_0^{2\pi} \tfrac{1}{2}(1 + \cos 2nx)\, dx = \pi \neq 0$.

7. Show that for $f \in L^1(\mathbb{R}^n)$,

$$\lim_{y\to 0} \int_{\mathbb{R}^n} |f(x) - f(x-y)|\, dx = 0 .$$

(10)

Let $\varepsilon > 0$.
Since $C_c(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$, there exists $g \in C_c(\mathbb{R}^n)$ s.t.

$$\|g - f\|_{L^1} < \frac{\varepsilon}{3} .$$

$$\Rightarrow \int_{\mathbb{R}^n} |f(x) - f(x-y)|\, dx \le \underbrace{\int_{\mathbb{R}^n} |f(x) - g(x)|\, dx}_{<\frac{\varepsilon}{3}} + \int_{\mathbb{R}^n} |g(x) - g(x-y)|\, dx + \underbrace{\int_{\mathbb{R}^n} |g(x-y) - f(x-y)|\, dx}_{<\frac{\varepsilon}{3}}$$

Since g is uniformly continuous on its compact support, we can choose y small enough s.t.

$$|g(x) - g(x-y)| < \frac{\varepsilon}{3}\, \mu(\operatorname{supp} g)^{-1} .$$

Then $\displaystyle\int_{\mathbb{R}^n} |g(x) - g(x-y)|\, dx < \frac{\varepsilon}{3}$.

$$\Rightarrow \int_{\mathbb{R}^n} |f(x) - f(x-y)|\, dx < \varepsilon .$$

$\square$

8. Compute the Fourier transform of the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x\,e^{-x^2}.$$

Hint: Recall from class that

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} f(x)\,dx;$$

when $g(x) = e^{-x^2}$, then

$$\hat{g}(\xi) = \frac{1}{\sqrt{2}} e^{-\xi^2/4}.$$

(10)

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x}\, x\, e^{-x^2}\, dx = \frac{1}{\sqrt{2\pi}} \frac{1}{-i} \frac{d}{d\xi} \int_{\mathbb{R}} e^{-i\xi x}\, e^{-x^2}\, dx$$

$$= i \frac{1}{\sqrt{2}} \frac{d}{d\xi} e^{-\frac{\xi^2}{4}}$$

$$= \frac{-i}{2^{3/2}} \xi\, e^{-\frac{\xi^2}{4}}$$

9. Let $T \in \mathcal{D}'(\mathbb{R})$ such that $xT = 0$. Prove that T must be a multiple of the δ-distribution.

Hint: Choose two different test functions $\phi, \psi \in \mathcal{D}(\mathbb{R})$ and apply T to $\theta = \phi/\phi(0) - \psi/\psi(0)$. (10)

Following the hint, note that $\theta(0) = 0$, so that

$$\theta = x\,\tilde{\theta} \qquad \text{with } \tilde{\theta} \in \mathcal{D}(\mathbb{R}).$$

$$\Rightarrow \quad T(\theta) = T(x\tilde{\theta}) \equiv (xT)(\tilde{\theta}) = 0 \quad \text{by assumption}$$

$$\Rightarrow \quad \frac{T(\phi)}{\phi(0)} = \frac{T(\psi)}{\psi(0)}$$

Since the LHS does not depend on ψ, the RHS cannot depend on ψ, i.e. it must be a constant (only depending on T). I.e.,

$$\frac{T(\psi)}{\psi(0)} = c \quad \Rightarrow \quad T(\psi) = c\,\psi(0).$$

$\Rightarrow$ T is a multiple of the δ-distribution.