Real Analysis

Homework 9

due November 12, 2008

1. Assume that (Ω, Σ, μ) is a measure space, where Σ is generated by an algebra $\mathcal{A}$. Assume that the measure space is σ-finite in the strong sense that
$$\Omega = \bigcup_{j=1}^{\infty} A_j$$
with $A_j \subset \mathcal{A}$ and $\mu(A_j) < \infty$. Let $f \in L^p(\Omega)$ for $1 \leq p < \infty$.

   Show that for every $\varepsilon > 0$ there exists a really simple function g such that
$$\|f - g\|_p < \varepsilon \, .$$

   Note: The case $p = 1$ was proved in class. To reduce the general case to the case $p = 1$, you may refer to Question 1 from Homework 8.

2. Show that if $f \in L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$ and $g \in C_c^\infty(\mathbb{R}^n)$, then $f * g \in C^\infty(\mathbb{R}^n)$ and for every multi-index α,
$$D^\alpha(f * g) = (D^\alpha g) * f \, .$$

3. Let H be a Hilbert space and $K \subset H$ be closed and convex. Then for every $x \in H \setminus K$ there exists $y \in K$ such that
$$\|x - y\| = \operatorname{dist}(x, K) \, .$$
   Moreover, for every $z \in K$,
$$\operatorname{Re}\langle z - y, x - y\rangle \leq 0 \, .$$

   Note: This is the Hilbert space version of Lieb and Loss, Lemma 2.8 (Projection onto convex sets).

4. Show that every Hilbert space is dual to itself with the obvious identification that $x \in H$ gives rise to $\ell \in H^*$ via $\ell(y) = \langle x, y\rangle$ for every $y \in H$.

   Note: This is the Hilbert space version of Lieb and Loss, Theorem 2.14 (Dual of L^p). The proof is analogous and needs the result from Question 3 above.