Real Analysis

Homework 6

due October 15, 2008

1. (From Folland.) Suppose $0 < p_0 < p_1 \leq \infty$. Find examples of functions f on $(0, \infty)$ endowed with the Lebesgue measure such that $f \in L^p$ if and only if

   (a) $p_0 < p < p_1$

   (b) $p_0 \leq p \leq p_1$

   (c) $p = p_0$

   Hint: Consider functions of the form $x^{-a} \, |\ln x|^b$.

2. (From Folland.) If $1 \leq p < r \leq \infty$, show that $L^p \cap L^r$ is a Banach space with norm $\|f\| = \|f\|_p + \|f\|_r$, and if $p < q < r$, the inclusion map $\mathrm{id} \colon L^p \cap L^r \to L^q$ is continuous.

3. Write up the proof of the uniform boundedness principle (cf. Lieb and Loss, Theorem 2.12) for the case $p = \infty$.

4. Let $f_k(x) = k^{\frac{1}{p}} \, g(kx)$ for some fixed $g \in L^p(\mathbb{R})$ with $1 < p < \infty$. Show that f_k converges weakly, but not strongly to zero in $L^p(\mathbb{R})$.