Real Analysis

Homework 5

due October 8, 2008

1. (From Folland.) Let
$$f(x,y) = (1-xy)^{-a}$$
with $a > 0$. Investigate the existence and equality of
$$\int_{[0,1]\times[0,1]} f \,\mathrm{d}(x,y)\,, \qquad \int_0^1 \int_0^1 f(x,y)\,\mathrm{d}x\,\mathrm{d}y\,, \qquad \text{and} \qquad \int_0^1 \int_0^1 f(x,y)\,\mathrm{d}y\,\mathrm{d}x\,.$$

2. (From Folland.) Show that
$$\int_0^\infty \mathrm{e}^{-sx}\, x^{-1}\, \sin^2 x\,\mathrm{d}x = \frac{1}{4}\,\ln(1+4s^{-2})$$
for $s > 0$ by integrating $\mathrm{e}^{-sx}\,\sin(2xy)$ with respect to x and y.

3. (From Lieb and Loss.) Let (Ω, Σ, μ) be a finite measure space, let f and $\{f_j\}_{j\in\mathbb{N}}$ be complex-valued, measurable functions on Ω with
$$\lim_{j\to\infty} f_j(x) = f(x)$$
for almost every $x \in \Omega$, and assume that
$$\int_\Omega |f_j|^2\,\mathrm{d}\mu < 1$$
and
$$\int_\Omega |f|^2\,\mathrm{d}\mu < \infty\,.$$
   (a) Prove that
   $$\int_\Omega |f_j - f|^p\,\mathrm{d}\mu \to 0$$
   as $j \to \infty$ for any $0 < p < 2$.

   (b) Construct a counter-example to show that this can fail for $p = 2$.

4. Let (Ω, Σ, μ) be a measure space and $f \in L^\infty(\Omega) \cap L^q(\Omega)$ for some $q \in [1, \infty)$. Then $f \in L^p(\Omega)$ for all $p > q$ and
$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p\,.$$