Real Analysis

Homework 3

due September 24, 2008

1. Let μ denote the Lebesgue measure on $\mathbb{R}$. Give a self-contained proof of the *outer regularity* of μ, i.e., show that

$$\mu(A) = \inf\{\mu(E)\colon E \text{ open and } A \subset E\}\,.$$

2. (From Lieb and Loss.)

   (a) Show that $F\colon \Omega \to \mathbb{R}$ is measurable if and only if $\{x \in \Omega\colon f(x) > a\}$ is measurable for every $a \in \mathbb{Q}$.

   (b) For $a \in \mathbb{Q}$ and f, g measurable, show that

$$\{x \in \Omega\colon f(x) + g(x) > a\} = \bigcup_{b \text{ rational}} \{x \in \Omega\colon f(x) > b\} \cap \{x \in \Omega\colon g(x) > a - b\}\,.$$

   Conclude that $f + g$ measurable.

   (c) Similarly, show that fg measurable if f and g are measurable.

3. (From Lieb and Loss.) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f(x) = |x|^{-p}\,\chi_{\{|x|<1\}}(x)\,.$$

   Compute the integral of f over $\mathbb{R}^n$ in two ways:

   (a) Use polar coordinates and compute the integral by the standard rules of calculus.

   (b) Compute $\mathcal{L}^n(\{x \in \mathbb{R}^n\colon f(x) > a\})$ and use the definition of the Lebesgue integral.

4. Prove that a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is continuous if and only if it is *upper semicontinuous* (the set $\{x\colon f(x) < t\}$ is open for all t) and *lower semicontinuous* (the set $\{x\colon f(x) > t\}$ is open for all t).