Real Analysis

Homework 12

due December 3, 2008

1. Show that $\{e^{ikx} \colon k \in \mathbb{Z}\}$ separates points on $\mathbb{T}$.

2. Let $\Omega \subset \mathbb{R}^n$ be open and μ a (positive) Borel measure such that $\mu(K) < \infty$ for all compact $K \subset \Omega$. Show that
$$T_\mu(\phi) = \int_\Omega \phi \, d\mu$$
defines a distribution $T_\mu \in \mathcal{D}'(\Omega)$.

3. Let $\Omega \subset \mathbb{R}^n$ be open. Let f_j be a sequence in $W^{1,1}_{\text{loc}}(\Omega)$, the space of $L^1_{\text{loc}}(\Omega)$ functions whose first order distributional derivatives are also $L^1_{\text{loc}}(\Omega)$, such that
$$f_j \to f \qquad \text{in } L^1_{\text{loc}}(\Omega)$$
and, for fixed $i \in 1, \dots, n$,
$$\partial_i f_j \to g \qquad \text{in } L^1_{\text{loc}}(\Omega)$$
where $\partial_i f$ denotes the distributional partial derivative in the ith coordinate direction. Show that
$$\partial_i f = g \, .$$
Hint: See comments in Lieb and Loss, Section 6.7.

4. Let $j \in L^1(\mathbb{R}^n)$ with
$$\int_{\mathbb{R}^n} j \, dx = 1 \, ,$$
and set
$$j_\varepsilon(x) = \frac{1}{\varepsilon^n} \, j\Big(\frac{x}{\varepsilon}\Big) \, .$$
Show that
$$j_\varepsilon \to \delta \qquad \text{in } \mathcal{D}'(\mathbb{R}^n)$$
as $\varepsilon \to 0$ (where j_ϵ is identified with a distribution in the canonical way).