Real Analysis

Midterm Exam

October 22, 2008

1. Let (Ω, Σ, μ) be a measure space and $E \in \Sigma$. Show that

$$\mu_E(A) = \mu(A \cap E) \qquad \text{for all } A \in \Sigma$$

defines a measure. (10)

2. Show that a monotonic function on $\mathbb{R}$ is Borel-measurable. (10)

3. Let (Ω, Σ, μ) be a measure space and $f\colon \Omega \to [0, \infty]$ a measurable function. Show that, for $0 < p < \infty$,

$$\int_\Omega f^p \, d\mu = p \int_0^\infty t^{p-1} \, \mu(\{x \in \Omega\colon f(x) > t\}) \, dt \, .$$

(10)

4. *Theorem on differentiation under the integral.*

Let (Ω, Σ, μ) be a measure space and let $I \subset \mathbb{R}$ open. Suppose that $f\colon \Omega \times I \to \mathbb{R}$ satisfies

(i) $f(\cdot, t)$ is measurable for every fixed $t \in I$;

(ii) $f(x, \cdot)$ is differentiable for almost every fixed $x \in \Omega$;

(iii) $|\partial f(x,t)/\partial t| \le g(x)$ for some $g \in L^1(\Omega)$.

Then

$$\frac{d}{dt} \int_\Omega f(x,t) \, d\mu(x) = \int_\Omega \frac{\partial f(x,t)}{\partial t} \, d\mu(x)$$

for every $t \in I$.

(a) Prove the theorem.

(b) Let

$$I(t) = \int_0^\infty \frac{\sin x}{x} e^{-tx}\, dx\,.$$

Verify that you may differentiate under the integral, then compute $I'(t)$.

(c) Use the result from (b) to show that the improper integral

$$\int_0^\infty \frac{\sin x}{x}\, dx = \arctan(\infty) = \frac{\pi}{2}\,.$$

(10+10+10)

5. Let (Ω, Σ, μ) be a finite measure space.

   Show that for $1 \le p \le r \le \infty$, $L^p(\Omega) \supset L^r(\Omega)$. (10)

6. Let (Ω, Σ, μ) be a measure space, $1 < p \le \infty$ and $1/p + 1/q = 1$. Let $\ell \in L^p(\Omega)^*$ be represented by a function $g \in L^q(\Omega)$ in the sense that

$$\ell(f) = \int_\Omega f\, g\, d\mu$$

   for every $f \in L^p(\Omega)$. Show that

$$||\ell||_{L^p(\Omega)^*} = ||g||_{L^q(\Omega)}\,.$$

(10)

7. (a) Find a sequence of bounded, Lebesgue-measurable sets in $\mathbb{R}$ whose characteristic functions converge weakly in $L^2(\mathbb{R})$ to a function f with the property that $2f$ is a characteristic function of a set with positive measure.

   Note: Full credit if you verify that your example has the requested properties on some (nontrivial) subspace of L^2.

   (b) How about the possibility that $f/2$ is a characteristic function?

(10+10)