Real Analysis

Final Exam

December 19, 2008

1. A "totally unlucky" number is one that contains no seven in any decimal expansion. Compute the Lebesgue measure of the totally unlucky numbers in $[0, 1]$.[1] (10)

2. Let (Ω, Σ, μ) be a measure space, $1 \leq p < \infty$, and $f \colon \Omega \to [0, \infty]$ a measurable function. Show that
$$\mu\{x \in \Omega \colon f(x) > \alpha\} \leq \frac{1}{\alpha^p} \int_\Omega f^p \, d\mu \, .$$
(10)

3. Let (Ω, Σ, μ) be a measure space and $f \colon \Omega \to [0, \infty]$ a measurable function with
$$0 < \int_\Omega f \, d\mu < \infty \, .$$
Find, with justification,
$$\lim_{n \to \infty} \int_\Omega n \, \ln\left(1 + \left(\frac{f}{n}\right)^\alpha\right) d\mu$$
when
   (a) $\alpha = 1$,
   (b) $0 < \alpha < 1$,
   (c) $\alpha > 1$.

   (5+5+5)

4. Let $1 < p < \infty$. For $f \in L^p((0, \infty))$, which for simplicity you may assume nonnegative, define the local average function
$$F(x) = \frac{1}{x} \int_0^x f(\xi) \, d\xi \, .$$

[1] From: http://www.math.nthu.edu.tw/~dhtsai/real-analysis-sample-exam-with-solution-2007-11-12.pdf

Show that $F \in L^p((0,\infty))$ with

$$\|F\|_p \le \frac{p}{p-1}\,\|f\|_p\,.$$

Hint: Find a differential equation for F, then multiply by F^{p-1} and integrate. You may assume first that $f \in C_c^\infty((0,\infty))$, then extend by density.[2] (10)

5. Let H be a Hilbert space and $V \subset H$ a closed subspace. In linear algebra, the quotient space H/V is defined as the space of cosets $\{x + V : v \in V\}$. (In other words, it is the set of equivalence classes with respect to the equivalence relation $x \sim y$ if $x - y \in V$.)

   How can you define an inner product on H/V so that the *quotient map* $p\colon H \to H/V$ defined by $p(x) = x + V$ is continuous? (10)

6. Let H be a Hilbert space, $\{x_n\}_{n\in\mathbb{N}} \subset H$ and $x \in H$ such that

$$x_n \rightharpoonup x$$

   weakly.

   (a) Show that

$$\|x\| \le \liminf_{n\to\infty} \|x_n\|\,.$$

   (b) Give an example where (a) holds with strict inequality.

   (10+5)

7. Show that for $f \in L^1(\mathbb{R}^n)$,

$$\lim_{y\to 0} \int_{\mathbb{R}^n} |f(x) - f(x-y)|\,dx = 0\,.$$

   (10)

8. Compute the Fourier transform of the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x\,e^{-x^2}\,.$$

   Hint: Recall from class that

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x}\, f(x)\,dx\,;$$

   when $g(x) = e^{-x^2}$, then

$$\hat{g}(\xi) = \frac{1}{\sqrt{2}}\, e^{-\xi^2/4}\,.$$

   (10)

[2] From: http://www.math.umn.edu/~lewicka/8601-2/ex19.pdf

9. Let $T \in \mathcal{D}'(\mathbb{R})$ such that $xT = 0$. Prove that T must be a multiple of the δ-distribution.

Hint: Choose two different test functions $\phi, \psi \in \mathcal{D}(\mathbb{R})$ and apply T to $\theta = \phi/\phi(0) - \psi/\psi(0)$. (10)