1. Solve the partial differential equation

$$x\, u_t + t\, u_x = 0\,,$$

where $u = u(x,t)$ for $(x,t) \in \mathbb{R}^2 \setminus (0,0)$. What are the characteristic curves? What kind of initial/boundary data do you need to prescribe?

Hint: Start out with an ansatz for the charactistic curves of the general form $x = x(s)$ and $t = t(s)$. (10)

Let $z(s) = u(x(s), t(s))$

$\Rightarrow z'(s) = u_x\,(x(s),t(s))\, x'(s) + u_t\,(x(s),t(s))\, t'(s)$

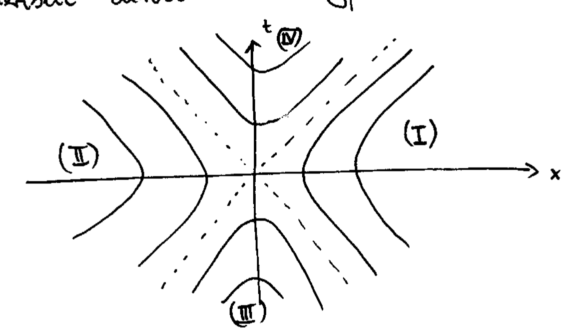
$z'(s) = 0$ provided $x' = \frac{1}{x}$, $t' = \frac{1}{t}$ (*)

Want $x(0) = x_0$, $x(s) = x$
$t(0) = 0$, $t(s) = t$,

so we can solve characteristic ODEs (*) with these conditions:

$$\left.\begin{array}{l} \displaystyle\int_{x_0}^{x(s)} x\,dx = \int_0^s ds \Rightarrow x^2 - x_0^2 = 2s \\ \displaystyle\int_0^{t(s)} t\,dt = \int_0^s ds \Rightarrow t^2 = 2s \end{array}\right\} \; x^2 - x_0^2 = t^2$$

$\Rightarrow$ The characteristic curves are the hyperbolae $x^2 - t^2 = x_0^2$:



$\Rightarrow$ Need to prescribe data on all axes.

In Sector I, for example,

$$u(x,t) = g\left(\sqrt{x^2 - t^2}\right) \qquad (**)$$

where g is the given data on the positive x-axis, with similar expressions in all other sectors. (In sectors (III) and (IV) need to solve ODEs with $x(0) = 0$, $t(0) = t_0$ instead.)

In principle, we could continuously extend the solution onto the lines $t = \pm x$ provided the data is compatible at the origin.

Also note that, strictly speaking, (*) are only valid for $x, t \neq 0$. However, it can be checked easily that the solutions (**) are valid throughout their sector as claimed.

Finally, note that for $x\, u_t - t\, u_x = 0$, the characteristic curves would have been circles, so we'd only need to prescribe data on the positive x-axis (for example), and we'd need to exclude the origin as claimed.

2. Show that if u is harmonic on some open $U \subset \mathbb{R}^n$, then u cannot have an isolated zero in U. (5)

Suppose u has an isolated zero at $x \in U$. Then $u \not\equiv$ const on any $B(x,r) \subset U$.

Since

$$u(x) = \fint_{\partial B(x,r)} u(y)\, dS(y) = 0 \,,$$

u must be positive and negative somewhere on $\partial B(x,r)$, by continuity, it must also have a zero on $\partial B(x,r)$.

Since $B(x,r) \subset \bar{U}$ is arbitrary, the zero at x cannot be isolated.

3. Let $U \in \mathbb{R}^n$ be open and bounded. State a set of boundary conditions which are sufficient to guarantee that a solution $u \in C^4(\bar{U})$ of the Poisson-type problem for the bi-Laplacian,

$$\Delta^2 u = f \,, \qquad (*)$$

satisfying those boundary conditions, is unique.

Hint: Energy methods. (10)

Assume u, v solve $(*)$, then $w = u - v$ solves

$$\Delta^2 w = 0 \qquad \text{in } U$$

$$\Rightarrow \quad 0 = \int_U w \Delta^2 w \, dx = \int_{\partial U} w \, \nu \cdot D\Delta w \, dS - \int_U Dw \cdot D\Delta w \, dx$$

$$= \int_{\partial U} w \, \nu \cdot D\Delta w \, dS - \int_{\partial U} \nu \cdot Dw \, \Delta w \, dS + \int_U (\Delta w)^2 \, dx$$

So if we specify u and either $\nu \cdot Du$ or Δu on the boundary, the first two terms drop out and we conclude

$$\begin{aligned} \Delta w &= 0 \quad \text{in } U \\ w &= 0 \quad \text{on } \partial U \,. \end{aligned}$$

By the uniqueness of the solution to the Poisson equation, $w = 0$ follows readily, implying uniqueness of the solution to $(*$

4. Consider the following initial-boundary value problem (IBVP) for the heat equation:

$$\begin{aligned} u_t - u_{xx} &= 0 && \text{in } (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \times (0,\infty)\,, \\ u &= 0 && \text{on } \{x = \pm\tfrac{\pi}{2}\} \times (0,\infty)\,, \\ u &= g && \text{on } [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \times \{t = 0\}\,. \end{aligned}$$

(a) Let $u_i \in C_1^2((-\frac{\pi}{2}, \frac{\pi}{2}) \times (0,\infty))$ solve the IBVP with initial data $g_i \in C([-\frac{\pi}{2}, \frac{\pi}{2}])$ for $i = 1, 2$. Show that if $g_1 \le g_2$, then $u_1 \le u_2$ for all $(x,t) \in [-\frac{\pi}{2}, \frac{\pi}{2}] \times [0,\infty)$.
Note: You may quote a well-known theorem from class; no need to prove from scratch.

(b) Show that the IBVP has a particular solutions of the form

$$u(x,t) = a(t)\cos x\,.$$

Derive an expression for $a(t)$.

(c) Conclude that

$$|u(x,t)| \le c\,e^{-t}$$

where c depends only on g.
Note: For simplicity of the argument, you may assume that g is compactly supported in $(\frac{\pi}{2}, \frac{\pi}{2})$.

(5+5+5)

(a) $g_1 - g_2 \le 0$

$\Rightarrow \max\limits_{\Gamma_T} (u_1 - u_2) = 0$ for any $T > 0$

Thus, by maximum principle for the heat equation (which is satisfied by $u_1 - u_2$),

$$\max_{U_T} (u_1 - u_2) = 0$$

$\Rightarrow u_1 \le u_2$

(b) Clearly, $u(x,t) = 0$ on $\{x = \pm\frac{\pi}{2}\} \times (0,\infty)$.

Moreover,

$$u_t = a'(t)\cos x$$
$$u_{xx} = a(t)(-\cos x)$$

So for $u_t - u_{xx} = 0$, need $a' = -a$

$\Rightarrow a(t) = c\,e^{-t}$ for any $c \in \mathbb{R}$.

(c) If g has compact support, then we can choose A large enough such that

$$g(x) \le c\cos x \quad \text{for } x \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2})$$

$\overset{(a)}{\Longrightarrow} \quad u(x,t) \le c\,e^{-t}\cos x \le c\,e^{-t}$

The claim follows with the same argument applied to $-g$.

5. Let $f \in C^2(\mathbb{R})$ and $b \in \mathbb{R}^n$. Show that $f(b \cdot x - t)$ solves the wave equation in $\mathbb{R}^n$. Describe the geometry of the solution. What is the speed of propagation? (10)

$$u(x,t) = f(b \cdot x - t)$$

$$\Rightarrow \quad u_{tt} = f''(b \cdot x - t)$$

$$Du = b\, f'(b \cdot x - t)$$

$$\Delta u = D \cdot Du = |b|^2 f''(b \cdot x - t)$$

$$\Rightarrow \quad u_{tt} - \frac{1}{|b|^2} \Delta u = 0 \qquad \text{(wave equation)}$$

Clearly, u is spatially constant on hyperplanes perpendicular to b. Thus, any feature of the solution must propergate in the direction of b.

To find the speed of propagation, let $s(t)$ "track a feature", i.e. be such that the argument of f is constant:

$$|b|\, s(t) - t = \text{const}$$

$$\Rightarrow \quad |b|\, \dot{s} - 1 = 0$$

$$\Rightarrow \quad \dot{s} = \frac{1}{|b|} \qquad \text{(wave speed)}.$$