

# Partial Differential Equations

## Homework 7

due November 12, 2008

In the following,  $\mathbb{T}$  denotes the 1-torus, i.e.  $\mathbb{T} = \mathbb{R} \bmod 2\pi$ .

1. (a) Consider a sequence  $u_n \in L^2$  with  $u_n \rightharpoonup u \in L^2$  weakly. Show that

$$\|u\| \leq \liminf_{n \rightarrow \infty} \|u_n\|. \quad (*)$$

(Remark: This statement is actually true for any Banach space, but the proof is not so elementary.)

- (b) Give an example where (\*) holds with strict inequality.

2. Prove that  $C(\mathbb{T}) \supset H^1(\mathbb{T})$ , and that the embedding is continuous; in particular, that there exists a constant  $c$  such that

$$\sup_{x \in \mathbb{T}} |u(x)| \leq c \|u\|_{H^1}.$$

3. (a) Show that, for every  $u \in L^r(\mathbb{T})$  with  $2 \leq r < \infty$ ,

$$\|u\|_{L^2} \leq (2\pi)^{\frac{r-2}{2r}} \|u\|_{L^r}.$$

*Hint:* Hölder inequality.

- (b) Consider the Fisher–Kolmogorov equation on  $\mathbb{T}$ ,

$$\begin{aligned} u_t &= u_{xx} + (1 - u)u^m, \\ u(0) &= u^{\text{in}}, \end{aligned}$$

where  $m$  is an even positive integer. Use the result from (a) to sharpen the  $L^2$  estimate derived in the lecture as follows: Show that

$$\limsup_{t \rightarrow \infty} \|u(t)\|_{L^2} \leq C$$

where an explicit estimate for  $C$  can be given which, in particular, shows that  $C$  does not depend on the initial data  $u^{\text{in}}$ .

4. Show that if  $u^{\text{in}} \geq 0$ , the solution  $u(t)$  to the Fisher–Kolmogorov equation remains nonnegative for every  $t \geq 0$ . You may assume that  $u$  is as smooth as you need.

*Hint:* This is similar to Homework 6, Question 1.