Partial Differential Equations

Homework 5

due October 7, 2008

1. Finish the proof of the solution formula for the inhomogeneous heat equation (Evans, p. 50, Theorem 2) by showing that for $f \in C(\mathbb{R}^n \times [0,\infty)) \cap L^\infty \mathbb{R}^n \times [0,\infty))$,

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \Phi(y,\varepsilon)\, f(x-y, t-\varepsilon)\, dy = f(x,t)$$

for $x \in \mathbb{R}^n$ and $t > 0$, where Φ denotes the fundamental solution of the heat equation.

Hint: Follow the the steps in the proof of the solution theorem for the initial value problem of the heat equation (Evans, p. 47, Theorem 1,iii).

2. Finish the proof of the mean value formula for the heat equation by showing that

$$\iint_{E(0,0;1)} \frac{|y|^2}{s^2}\, dy\, ds = 4\,,$$

where

$$E(x,t;r) = \{(y,s) \colon \Phi(x-y, t-s) \geq r^{-n}\}$$

denotes the heat ball "centered" at (x,t).

Hint: Use polar coordinates in space, and an appropriate change of variables in time. The remaining one-dimensional integral is Mathematica-integrable. You can also use that

$$\int_0^\infty t^{\lambda+1}\, e^{-\lambda t}\, dt = \frac{\Gamma(\lambda+2)}{\lambda^{2+\lambda}}\,,$$

$$\Gamma(x+1) = x\,\Gamma(x)\,,$$

$$\alpha(n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}\,.$$

3. Evans, p. 87 problem 13