Partial Differential Equations

Final Exam

December 11, 2008

1. (a) Fix $x_0 \in \mathbb{R}^n$ with $n \geq 2$ and set $R = |x_0|$. Show that

$$h(x) = \frac{|x|^2 - R^2}{|x - x_0|^n} + R^{2-n}$$

is a harmonic function on $\mathbb{R}^n \setminus \{x_0\}$ and that $h(0) = 0$.

(b) *Kuran's theorem.* Let $U \subset \mathbb{R}^n$ be an open, bounded, and connected set which contains the origin. Suppose that

$$\fint_U u(x)\,dx = u(0)$$

for every function u which is harmonic on U. Prove that U must be a ball centered at the origin.

Hint: Use the function from (a) with a clever choice of R and x_0 to test the mean value property.

(10+10)

2. Let $U \subset \mathbb{R}^n$ be open and bounded and fix $T > 0$. Recall the definition of the parabolic cylinder

$$U_T \equiv U \times (0, T]$$

and the parabolic boundary

$$\Gamma_T \equiv \overline{U}_T \setminus U_T\,.$$

Show that there exists at most one solution $u \in C^2_1(\overline{U}_T)$ to the reaction diffusion equation

$$\begin{aligned} u_t &= \Delta u - u^2 \qquad \text{in } U_T\,, \\ u &= g \qquad \text{on } \Gamma_T\,. \end{aligned}$$

(10)

3. (a) Show that the solution to the wave equation

$$\begin{aligned} u_{tt} - u_{xx} &= 0 \qquad \text{in } \mathbb{R} \times (0, \infty)\,, \\ u &= g \qquad \text{on } \mathbb{R} \times \{t = 0\}\,, \\ u_t &= h \qquad \text{on } \mathbb{R} \times \{t = 0\} \end{aligned}$$

for $g \in C^2$ and $h \in C^1$ is given by

$$u(x,t) = \frac{g(x+t) + g(x-t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} h(y)\, dy\,.$$

What is the speed of propagation?

(b) Let $U = [a, b]$ and define U_T and Γ_T as in Question 2.
Show that $u \in C^2(\overline{U}_T)$ solves the wave equation in U_T if and only if

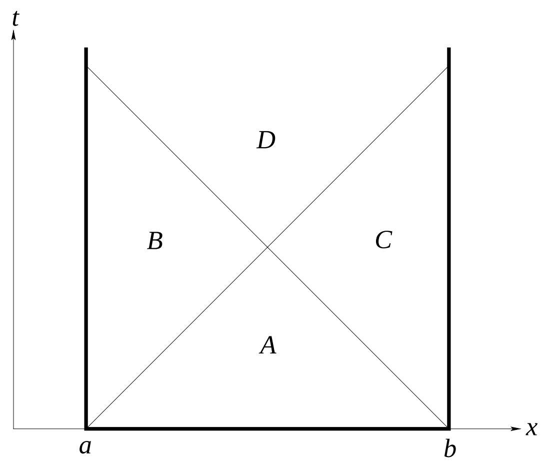
$$u(x - \xi, t - \tau) + u(x + \xi, t + \tau) = u(x + \tau, t + \xi) + u(x - \tau, t - \xi)$$

for all $\xi, \tau \geq 0$ such that the function arguments are contained in $\overline{U}_T$.
Hint: The "only if" part is difficult. One way of proving it starts from the factorization of the wave equation $u_{tt} - u_{xx} = (\partial_t - \partial_x)(\partial_t + \partial_x)u = 0$ used in class for the constructive derivation of the solution formula from (a).

(c) Prove the existence of a solution to the initial-boundary value problem

$$\begin{aligned} u_{tt} - u_{xx} &= 0 \qquad \text{in } U_T\,, \\ u &= g \qquad \text{on } \Gamma_T\,, \\ u_t &= h \qquad \text{on } U \times \{t = 0\}\,. \end{aligned}$$

Hint: No computations are necessary. Think about how to construct the solutions using (a) and (b) on different segments of the x-t plane:



(10+10+10)

4. Let u be a weak solution of Burgers' equation

$$\begin{aligned} u_t + u\,u_x &= 0 \qquad \text{in } \mathbb{R} \times [0, \infty)\,, \\ u &= g \qquad \text{on } \mathbb{R} \times \{t = 0\}\,, \end{aligned}$$

where g is a smooth bounded function all of whose derivatives go to zero as $|x| \to \infty$.

Show that the characteristic curve which first runs into a shock is one which originates from a point of inflection of g at time $t = 0$.

Hint: Find an equation for u_x along a characteristic line. A shock is characterized by $|u_x| \to \infty$. (10)

5. Let $U \subset \mathbb{R}^n$ be an open set with sufficiently smooth boundary, and $u \colon U \to \mathbb{R}$ a smooth function with $u = 0$ on ∂U.

Show that

$$\int_U |Du|^2 \, dx \le \left(\int_U |u|^2 \, dx \right)^{\frac{1}{2}} \left(\int_U |\Delta u|^2 \, dx \right)^{\frac{1}{2}}.$$

(10)