1. Prove, by induction, that $2^{2n} - 1$ is divisible by 3 for every $n \in \mathbb{N}$. (8)

$\underline{n=1}$: $2^{2n} - 1 = 4 - 1 = 3$ is divisible by 3.

$\underline{n \to n+1}$: $2^{2(n+1)} - 1 = 4 \cdot 2^{2n} - 1$

Since, by induction hypothesis, $2^{2n}-1$ is divisible by 3, there exists $k \in \mathbb{N}$ such that $2^{2n} - 1 = 3k$.

$\Rightarrow 2^{2n} = 3k+1$

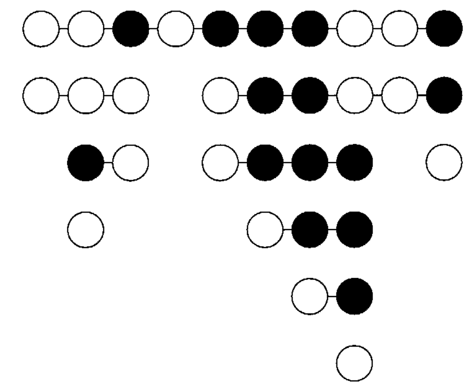
$\Rightarrow 2^{2(n+1)} - 1 = 4(3k+1) - 1 = 3(4k+1)$

which is clearly divisible by 3.

$\square$

2. Consider the game where n balls are arranged along a chain; k balls are white and the remaining n − k balls are black. You cannot remove a black ball. When removing a white ball, the chain breaks into two pieces and the former neighbors of the removed ball change their color.

It is sometimes, but not always, possible to remove all balls following these rules. The following example shows a case where it can be done.



(a) Conjecture a condition which ensures that all balls can be removed in accordance with the rules above.

(b) Prove, by induction or otherwise, that your condition is sufficient.

(c) Prove, by induction or otherwise, that your condition is necessary.

(3+3+3)

(a) Conjecture: all balls can be removed in accordance with the rules if and only if the number of white balls is odd.

(b) By induction: (assume the number of white balls is odd)

$n=1$: one white ball can always be removed

$1,\dots,n \to n+1$: Consider a chain of $n+1$ balls.

Remove the left-most white ball (always possible, since the number of white balls is odd).

We are left with two pieces, possibly empty, of at most n balls.

The left hand piece is either empty, or contains exactly one white ball (at its right hand end), hence can be removed by the induction hypothesis.

If k denotes the number of white balls in the original chain, then the right hand piece contains either k white balls (if the right hand neighbor of the removed ball was black) or $k-2$ white balls (if the right hand neighbor of the removed ball was white) – in any case, an odd number, so that the right hand piece can also be removed by induction hypothesis.

(c) By induction: (assume the number of white balls is even)

$n=2$: There are 0 or 2 white balls, which cannot all be removed.

$2,\dots,n \to n+1$: Consider a chain of $n+1$ balls.

Remove any one of k, with k even, white balls.

Then one of the remaining pieces has an even number of white balls, the other has an odd number of white balls. After changing the colors of the neighbors of the removed ball, we are still left with one piece with an even number of white balls (note that this piece is non-empty, even if the removed ball was at one end of the chain) and one piece with an odd number of white balls (or which is empty if the removed ball was at one end of the chain).

Thus, we are left with a non-empty chain of at most n balls of which an even number is white. This chain cannot be removed completely by induction hypothesis.

3. For $a, b \in \mathbb{Z}$, define $a \sim b$ if and only if $|a|^2 = |b|^2$.

Check whether this relation is an equivalence relation, i.e. check whether it is reflexive, symmetric, and transitive. If a property holds, prove that it does. If a property does not hold, give a counter example. (8)

- $|a|^2 = |a|^2 \Rightarrow a \sim a$, the relation is reflexive.
- $a \sim b \Rightarrow |a|^2 = |b|^2 \Rightarrow |b|^2 = |a|^2 \Rightarrow b \sim a$,

  the relation is symmetric.
- $$\left.\begin{array}{l} a \sim b \Rightarrow |a|^2 = |b|^2 \\ b \sim c \Rightarrow |b|^2 = |c|^2 \end{array}\right\} \Rightarrow |a|^2 = |c|^2 \Rightarrow a \sim c,$$

  the relation is transitive.

<u>Note</u>: All of the above holds, e.g., if $a, b \in \mathbb{C}$. In this case, the claims are slightly less trivial.

4. Let $(\mathbb{N}, s, 1)$ be a set with successor map s and distinguished element 1 satisfying the Peano axioms, and let $(\mathbb{N}', s', 1')$ be a different set with its own successor map s' and distinguished element $1'$ also satisfying the Peano axioms.

Assume that $\Phi: \mathbb{N} \to \mathbb{N}'$ is a map with $\Phi(1) = 1'$ and $\Phi(s(n)) = s'(\Phi(n))$ for every $n \in \mathbb{N}$. Prove that Φ is surjective. (8)

Let $M' = \{n' \in \mathbb{N}' : n' = \phi(n) \text{ for some } n \in \mathbb{N}\}$.

<u>$1' \in M'$</u>: true, since $\phi(1) = 1'$.

<u>$s'(M') \subset M'$</u>: Take $c' \in M'$

$\Rightarrow$ there exists $n \in \mathbb{N}$ s.t. $\phi(n) = c'$.

$$\Rightarrow s'(c') = s'(\phi(n)) = \phi(\underbrace{s(n)}_{\in \mathbb{N}})$$

$\Rightarrow s'(c') \in M'$

Thus, M' is inductive with respect to $(\mathbb{N}', s', 1') \Rightarrow M' = \mathbb{N}'$

This proves that ϕ is surjective.

5. In how many different ways can 5 boys and 5 girls sit in (a) a row and (b) a circle so that no boy sits next to a boy and no girl sits next to a girl? (6+3)

(a) The possible seating patterns are

BGBGBGBGBG

and GBGBGBGBGB .

Thus, we have

$$\underbrace{2 \cdot 5!}_{\text{number of ways of seating 5 boys in a row}} \cdot \overbrace{5!}^{\text{number of ways of seating 5 girls in a row}}$$

possibilities.

(c) Each of 10 possible cyclical shifts is considered equivalent, hence

$$\frac{2 \cdot 5! \cdot 5!}{10} = 4!\, 5!$$

possibilities.

6. Show that

$$\binom{n}{k}\binom{n-k}{j} = \binom{n}{j}\binom{n-j}{k}.$$

(8)

$$\binom{n}{k}\binom{n-k}{j} = \frac{n!}{k!\,(n-k)!}\;\frac{(n-k)!}{j!\,(n-k-j)!}$$

$$= \frac{n!}{j!\,(n-j)!}\;\frac{(n-j)!}{k!\,(n-k-j)!}$$

$$= \binom{n}{j}\binom{n-j}{k}$$