General Mathematics and Computational Science I

Final Exam

December 14, 2007

Recurrence relation for binomial coefficients:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Stirling's approximation:

$$n! \sim \sqrt{2\pi n}\, n^n \, e^{-n}$$

Taylor series of the logarithm:

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \dots$$

1. Let $f, g \colon \mathbb{N} \to \mathbb{N}$ such that their composition $h(n) = f(g(n))$ is injective.

   (a) Show that f is injective.

   (b) Give an example that shows that g need not be injective.

   (5+5)

2. Show that, for nonnegative $j \leq n$,

$$\sum_{k=j}^{n} \binom{k}{j} = \binom{n+1}{j+1} .$$

   (10)

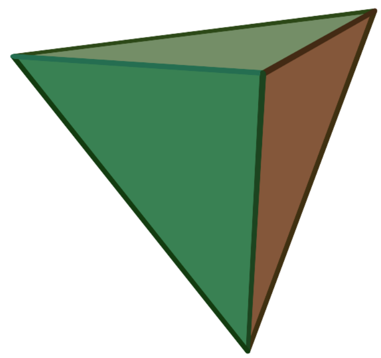
3. How many closed loops along the edges of a tetrahedron exist, provided that each edge is visited no more than once, and

   (a) all vertices (corners) of the tetrahedron are distinct and the orientation (the direction of travel along the loop) matters;

   (b) all vertices are distinct, but orientation does *not* matter;

(c) orientation matters, but vertices are not distinct? (In other words, loops are considered identical if they differ only by an arbitrary rotation of the tetrahedron in space.)

(5+5+5)



(Image of a tetrahedron from http://commons.wikimedia.org/wiki/Image:Tetrahedron.svg)

4. Consider n-words, i.e. words of length n, from the alphabet $\{A, B, C\}$.

   (a) Count the number of different n-words.

   (b) Count the number of different n-words which do not have neighboring As.

(5+5)

5. What is the probability that the product of n randomly chosen integers is odd? (10)

6. *(Birthday attack.)* Let $f, g \colon \mathbb{N} \to \{1, 2, \ldots, n\}$, where n is large. Assume that for a randomly chosen argument, each function yields any number from its range with equal probability.

   (a) Show that the probability that $f(i) = g(i)$ for at least one $i \in 1, \ldots, k$ is given by

   $$P = 1 - \frac{n!}{n^k \, (n-k)!} \, .$$

   (b) Assuming that $k \ll n$, show that

   $$P \approx 1 - e^{-\frac{k^2}{2n}} \, .$$

   Note: A rigorous estimate of the remainder term is not required, but you should explicitly state which approximations you make.

(c) Suppose your goal is to find a number i for which $f(i) = g(i)$, and you wish to succeed with probability $P = \frac{1}{2}$. How many different arguments will you have to try?

(5+5+5)

7. Show that for $a, b \geq 0$,

$$\sqrt{a+b} \leq \sqrt{a} + \sqrt{b} \leq \sqrt{2\,(a+b)}\,.$$

(10)

8. Find all equilibrium points on the open half-line $x > 0$ of the difference equation

$$x_{n+1} = x_n\,(1 - \ln x_n)$$

and determine their stability. (10)

9. Consider the difference equation

$$x_n + 2\,x_{n-1} + x_{n-2} = 4\,.$$

(a) Find the solution when $x_0 = 1$ and $x_1 = -1$.

(b) Does the difference equation have a stable equilibrium point?

(5+5)