Partial Differential Equations

Final Exam

Solutions

1. Consider the homogeneous Helmholtz equation on $\mathbb{R}^3$,

$$(1-\Delta)u = 0\,.$$

(a) Show that if u is a radial solution, i.e. $u(x) = v(r)$ with $r = |x|$, then

$$v - \frac{2}{r}v' - v'' = 0\,.$$

Solution.

$$Du(x) = v'(r)Dr = v'(r)\frac{x}{r},$$

$$\Delta u(x) = D\cdot Du(x) = D\cdot\left[v'(r)\frac{x}{r}\right] = v''(r) + v'(r)\frac{3}{r} - v'(r)\frac{x\cdot x}{r^3} = v''(r) + v'(r)\frac{2}{r},$$

$$0 = (1-\Delta)u(x) = v(r) - v''(r) - v'(r)\frac{2}{r}.$$

(b) Show that

$$v_\pm(r) = \frac{e^{\pm r}}{r}$$

are two (independent) radial solutions. Which one would you consider to use as a fundamental solution for the Helmholtz equation?

Solution.

$$v_\pm'(r) = -\frac{e^{\pm r}}{r^2} \pm \frac{e^{\pm r}}{r}, \qquad v''(r) = 2\frac{e^{\pm r}}{r^3} \mp \frac{e^{\pm r}}{r^2} \mp \frac{e^{\pm r}}{r^2} + \frac{e^{\pm r}}{r},$$

$$\frac{r}{e^{\pm r}}\left[v - \frac{2}{r}v' - v''\right] = 1 - \frac{2}{r}\left[-\frac{1}{r} \pm 1\right] - \frac{2}{r^2} \pm \frac{2}{r} - 1 \;= 1 + \frac{2}{r^2} \mp \frac{2}{r} - \frac{2}{r^2} \pm \frac{2}{r} - 1 = 0.$$

$v_-(r)$ vanishes at infinity and hence can be considered as a fundamental solution for the Helmholtz equation.

(10+10)

2. Let $U \subset \mathbb{R}^n$ be open and assume that $u \colon U \to \mathbb{R}$ is harmonic.

(a) Show that, for any $i = 1, \dots, n$,

$$|\partial_i u(x)| \leq \frac{n}{r} \|u\|_{L^\infty(\partial B(x,r))}$$

so long as $B(x,r) \subset U$.

Hint: Mean value formula.

Solution. The mean value formula reads

$$u(x) = \fint_{B(x,r)} u(y)dy = \fint_{\partial B(x,r)} u(y)dS(y) = \fint_{\partial B(0,1)} u(x+rz)dS(z) = \fint_{B(0,1)} u(x+rz)dz.$$

Then, since $\partial u_i(x)$ is also harmonic,

$$\partial_i u(x) = \fint_{B(x,r)} \partial_i u(y)dy = \frac{1}{\alpha(n)r^n} \int\limits_{B(x,r)} \partial_i u(y)dy = \frac{1}{\alpha(n)r^n} \int\limits_{\partial B(x,r)} u(y)dS(y).$$

Here we applied the Gauss formula. Then

$$|\partial_i u(x)| \leq \frac{\|u\|_{L^\infty(\partial B(x,r))}}{\alpha(n)r^n} \int\limits_{\partial B(x,r)} dS(y) = \|u\|_{L^\infty(\partial B(x,r))} \frac{n\alpha(n)r^{n-1}}{\alpha(n)r^n} = \frac{n}{r}\|u\|_{L^\infty(\partial B(x,r))}.$$

(b) Then conclude that

$$|\partial_i u(x)| \leq \frac{n}{\alpha(n)\, r^{n+1}} \|u\|_{L^1(B(x,2r))}$$

provided $B(x,2r) \subset U$.

Hint: Mean value formula.

Solution. First we note that if $y \in B(x,r)$, then $\forall z \in B(y,r) \Rightarrow z \in B(x,2r)$ due to the triangle inequality. Then

$$|u(y)| \leq \frac{1}{\alpha(n)r^n} \int\limits_{B(y,r)} |u(z)|\, dz \leq \frac{1}{\alpha(n)r^n}\|u\|_{L^1(B(x,2r))}, \quad \forall y \in B(x,r).$$

In particular, it holds for all $y \in \partial B(x,r)$ Therefore

$$\|u\|_{L^\infty(\partial B(x,r))} = \max_{y \in \partial B(x,r)} |u(y)| \leq \frac{1}{\alpha(n)r^n}\|u\|_{L^1(B(x,2r))}.$$

Result of 2a gives

$$|\partial_i u(x)| \leq \frac{n}{r}\|u\|_{L^\infty(\partial B(x,r))} \leq \frac{n}{\alpha(n)\, r^{n+1}} \|u\|_{L^1(B(x,2r))}.$$

(10+5)

3. Let $U \subset \mathbb{R}^n$ be open and bounded. Consider the Poisson equation with so-called Neumann boundary conditions,

$$\begin{aligned} -\Delta u &= f \quad \text{in } U\,, \\ \nu \cdot Du &= 0 \quad \text{on } \partial U\,. \end{aligned}$$

Show that this equation cannot have a solution unless

$$\int_U f \, dx = 0\,.$$

Solution.

$$\int\limits_U f \, dx = -\int\limits_U \Delta u \, dx = -\int\limits_{\partial U} \nu \cdot Du \, dS = 0.$$

(10)

4. A function $u \in L^1_{loc}(\mathbb{R} \times \mathbb{R})$ is called a *weak solution* of the wave equation on the line if

$$\int_{\mathbb{R}} \int_{\mathbb{R}} u(x,t) \left(\nu_{tt}(x,t) - \nu_{xx}(x,t)\right) dx \, dt = 0$$

for every $\nu \in C_0^\infty(\mathbb{R} \times \mathbb{R})$.

(a) Show that if $u \in C^2(\mathbb{R} \times \mathbb{R})$ is a classical solution of the wave equation, it is also a weak solution.

Solution.

$$\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} u(x,t) \left(\nu_{tt}(x,t) - \nu_{xx}(x,t)\right) dx \, dt = \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} u_t(x,t)\nu_t(x,t) - u_x(x,t)\nu_x(x,t) \, dx \, dt =$$

$$= \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} u_{tt}(x,t)\nu(x,t) - u_{xx}(x,t)\nu(x,t) \, dx \, dt = \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} (u_{tt}(x,t) - u_{xx}(x,t))\nu(x,t) \, dx \, dt = 0.$$

There are no boundary terms, since $\nu \in C_0^\infty(\mathbb{R} \times \mathbb{R})$.

(b) Verify, by explicit computation, that

$$u(x,t) = \begin{cases} 1 & \text{for } x > t \\ 0 & \text{for } x \le t \end{cases}$$

is a weak solution of the wave equation.

Solution.

$$\int_{\mathbb{R}}\int_{\mathbb{R}} u(x,t)\left(v_{tt}(x,t)-v_{xx}(x,t)\right)dx\,dt = \int_{\mathbb{R}}\int_{-\infty}^{x} v_{tt}(x,t)\,dt\,dx - \int_{\mathbb{R}}\int_{t}^{+\infty} v_{xx}(x,t)\,dx\,dt =$$

$$= \int_{\mathbb{R}} v_t(x,x)\,dx + \int_{\mathbb{R}} v_x(t,t)\,dt = \int_{\mathbb{R}}\left(v_t(y,y)+v_x(y,y)\right)dy = 0.$$

(5+10)

5. Consider the Korteweg–de Vries equation

$$u_t - 6uu_x + u_{xxx} = 0$$

on $\mathbb{T}\times(0,\infty)$. Show that

$$M = \int_{\mathbb{T}} u\,dx\,,$$
$$E = \int_{\mathbb{T}} u^2\,dx\,,$$

and

$$H = \int_{\mathbb{T}}\left(\tfrac{1}{2}u_x^2 + u^3\right)dx$$

are all constants of the motion.

Solution.

$$\frac{d}{dt}M = \int_{\mathbb{T}} u_t\,dx = \int_{\mathbb{T}}(6uu_x - u_{xxx})\,dx = (3u^2+u_{xx})\Big|_{\mathbb{T}} = 0,$$

$$\frac{1}{2}\frac{d}{dt}E = \int_{\mathbb{T}} uu_t\,dx = \int_{\mathbb{T}}\left(6u^2u_x - uu_{xxx}\right)dx = 2u^3\Big|_{\mathbb{T}} + \int_{\mathbb{T}} u_xu_{xx}\,dx = u_x^2\Big|_{\mathbb{T}} = 0,$$

$$\frac{d}{dt}H = \int_{\mathbb{T}}\left(u_xu_{xt}+3u^2u_t\right)dx = \int_{\mathbb{T}} u_t\left(u_{xx}+3u^2\right)dx = \int_{\mathbb{T}}(6uu_x-u_{xxx})\left(u_{xx}+3u^2\right)dx =$$

$$= \int_{\mathbb{T}}\left(6uu_xu_{xx} - u_{xxx}u_{xx} + 18u_xu^3 - 3u^2u_{xxx}\right)dx =$$

$$= \int_{\mathbb{T}}\left(6uu_xu_{xx} - [u_{xx}^2]_x + \frac{9}{2}[u^4]_x - 6uu_xu_{xx}\right)dx = \left[\frac{9}{2}u^4 - u_{xx}^2\right]\Bigg|_{\mathbb{T}} = 0.$$

There is no boundary terms, since $\mathbb{T}$ has no boundary.

(5+5+10)

6. Let H be a Hilbert space and u_n a sequence in H. Show that the following are equivalent.

(i) $u_n \to u$ strongly;

(ii) $u_n \rightharpoonup u$ weakly and $\|u_n\| \to \|u\|$.

Solution. "$\Rightarrow$" By definition,

$$u_n \to u \qquad \Leftrightarrow \qquad \|u_n - u\|_H \to 0.$$

Using the triangle inequality we conclude

$$\|u_n\|_H - \|u\|_H = \|u_n - u + u\|_H - \|u\|_H \leq \|u_n - u\|_H + \|u\|_H - \|u\|_H = \|u_n - u\|_H,$$

$$\|u_n\|_H - \|u\|_H = \|u_n\|_H - \|u - u_n + u_n\|_H \geq \|u_n\|_H - \|u - u_n\|_H - \|u_n\|_H = -\|u - u_n\|_H.$$

Hence,

$$\Big| \|u_n\|_H - \|u\|_H \Big| \leq \|u_n - u\|_H \to 0.$$

The Cauchy-Schwarz inequality gives

$$\Big| \langle u_n - u, w \rangle \Big| \leq \|u_n - u\|_H \, \|w\|_H \to 0 \qquad \forall w \in H.$$

Consequently,

$$\langle u_n, w \rangle \to \langle u, w \rangle \quad \forall w \in H \qquad \Leftrightarrow \qquad u_n \rightharpoonup u.$$

"$\Leftarrow$" By definition,

$$\langle u_n - u, w \rangle \to 0 \qquad \forall w \in H.$$

Take $w := u_n - u$. Then

$$\|u_n - u\|_H^2 = \langle u_n - u, u_n - u \rangle = \|u_n\|_H^2 - 2\langle u_n, u \rangle + \|u\|_H^2 \to \|u_n\|_H^2 - \|u\|_H^2.$$

Since the right hand side vanishes, we have $u_n \to u$ strongly.

(10+10)