Partial Differential Equations

Homework 9

due May 7, 2007

In the following, $\mathbb{T}$ denotes the 1-torus, i.e. $\mathbb{T} = \mathbb{R} \bmod 2\pi$.

1. Let $U \subset \mathbb{R}^n$ be open and let u_n and v_n be two sequences in $L^2(U)$.
   (a) Suppose $u_n \to u$ strongly and $v_n \rightharpoonup v$ weakly, and that, moreover, $u_n v_n \in L^2(U)$ for every $n \in \mathbb{N}$ and also $uv \in L^2(U)$.
   Show that $u_n v_n \rightharpoonup uv$ weakly.
   (b) Give an example that when $u_n \rightharpoonup u$ only weakly in (a), then $u_n v_n$ may not converge to uv weakly.

2. (a) Show that, for every $u \in H^2(\mathbb{T})$,
$$\|u\|_{H^1}^2 \le \|u\|_{L^2} \, \|u\|_{H^2} \, .$$
   (b) Consider the Fisher–Kolmogorov equation on $\mathbb{T}$,
$$u_t = u_{xx} + (1-u)\, u^m \, ,$$
$$u(0) = u^{\mathrm{in}} \, ,$$
   where m is an even positive integer. Use the result from (a), as well as Question 4 of the previous homework set, to prove that that
$$\limsup_{t \to \infty} \|u(t)\|_{H^1} \le C$$
   where an explicit estimate for C can be given which, in particular, shows that C does not depend on the initial data u^{in}. You may assume that u is sufficiently differentiable so that all your formal manipulations are justified.

3. Prove the following version of the *Poincaré inequality*: For every $u \in H^1(\mathbb{T})$ which has zero mean, i.e. where
$$⨍_{\mathbb{T}} u \, dx = 0 \, ,$$
   we have
$$\int_{\mathbb{T}} |u|^2 \, dx \le C \int_{\mathbb{T}} |u_x|^2 \, dx \, .$$
   Find the best estimate for C.

4. Consider the inviscid Burger's equation on $\mathbb{T}$, i.e.

$$u_t + u\, u_x = 0\,.$$

(a) Define an approximate solution u_n by applying the projector $\mathbb{P}_n$ which projects onto modes up to wave number n to Burger's equation. Show that

$$\|u_n(t)\|_{L^2} = \|u_n(0)\|_{L^2}\,.$$

(b) Conclude that $\{u_n\}$ has a subsequence that converges to some u weakly in $L^2(\mathbb{T})$, and that

$$\|u(t)\|_{L^2} \leq \|u(0)\|_{L^2}\,.$$

Why do you have an inequality rather than equality?

(Note: you are not required to show that u solves Burger's equation in any sense. This would require a much more involved analysis.)