Partial Differential Equations

Homework 8

due April 25, 2007

In the following, $\mathbb{T}$ denotes the 1-torus, i.e. $\mathbb{T} = \mathbb{R} \bmod 2\pi$.

1. Let $f, g \in L^1(\mathbb{R}^n)$, i.e.
$$\|f\|_{L^1} \equiv \int_{\mathbb{R}^n} |f(x)| \, dx < \infty \, ;$$
similarly for g. Show
   (a) $\displaystyle\lim_{y \to 0} \int_{\mathbb{R}^n} |f(x) - f(x-y)| \, dx = 0.$
   Hint: Use mollifiers.
   (b) $\|f * g\|_{L^1} \leq \|f\|_{L^1} \, \|g\|_{L^1}$
   (c) Suppose that, moreover, $g \in L^\infty(\mathbb{R}^n)$. Conclude that $f * g \in C(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$.

2. (a) Consider a sequence $u_n \in L^2$ with $u_n \rightharpoonup u \in L^2$ weakly. Show that
$$\|u\| \leq \liminf_{n \to \infty} \|u_n\| \, . \qquad (*)$$
   (Remark: This statement is actually true for any Banach space.)
   (b) Give an example where (*) holds with strict inequality.

3. Prove that $C(\mathbb{T}) \supset H^1(\mathbb{T})$.

4. (a) Show that, for every $u \in L^r(\mathbb{T})$ with $2 \leq r < \infty$,
$$\|u\|_{L^2} \leq (2\pi)^{\frac{r-2}{2r}} \, \|u\|_{L^r} \, .$$
   Hint: Hölder inequality.
   (b) Consider the Fisher–Kolmogorov equation on $\mathbb{T}$,
$$u_t = u_{xx} + (1-u) \, u^m \, ,$$
$$u(0) = u^{\text{in}} \, ,$$

where m is an even positive integer. Use the result from (a) to sharpen the L^2 estimate derived in the lecture as follows: Show that

$$\limsup_{t\to\infty} \|u(t)\|_{L^2} \le C$$

where an explicit estimate for C can be given which, in particular, shows that C does not depend on the initial data u^{in}.

5. Show that if $u^{\text{in}} \ge 0$, the solution $u(t)$ to the Fisher–Kolmogorov equation remains nonnegative for every $t \ge 0$. You may assume that u is as smooth as you need.

   Hint: This is similar to Homework 6, Question 2.