Partial Differential Equations

Homework 2

due February 21, 2007

1. (a) The *standard mollifier* is defined by

$$\eta(x) \equiv \begin{cases} c(n)\, \exp\left(\dfrac{1}{|x|^2 - 1}\right) & \text{if } |x| < 1 \\ 0 & \text{otherwise}\,, \end{cases}$$

where $c(n)$ is chosen such that

$$\int_{\mathbb{R}^n} \eta(x)\, dx = 1\,.$$

Show that $\eta \in C^\infty(\mathbb{R}^n)$.

(b) Show that if η_ε is a radial mollifier, and u is a radial, locally integrable function, then its mollification

$$u_\varepsilon(x) = (\eta_\varepsilon * u)(x) = \int_{\mathbb{R}^n} \eta_\varepsilon(y)\, u(x-y)\, dy$$

is also radial.

2. Let $X \subset \mathbb{R}^n$. Show that

(a) X is connected iff $\emptyset$ and X are the only subsets of X that are both relatively open and relatively closed in X.

(b) If $\{W_\alpha\}_{\alpha\in A}$ is a collection of connected subsets of X such that

$$\bigcap_{\alpha\in A} W_\alpha \neq \emptyset\,,$$

then $\cup_{\alpha\in A} W_\alpha$ is connected.

(c) If X is connected, then $\overline{X}$ is connected.

(d) Every point $x \in X$ is contained in a unique maximal connected subset of X, and this subset is relatively closed in X.

The relevant definitions from point-set topology in $\mathbb{R}^n$:

- $A \subset X$ is called *relatively open in* X if for every $x \in A$ there exists an $\varepsilon > 0$ such that $X \cap B(x, \varepsilon) \subset A$.
- $B \subset X$ is called *relatively closed in* X if $A = X \setminus B$ is relatively open in X.
- X is called *disconnected* if there exist disjoint, nonempty subsets $A_1, A_2 \subset X$ that are relatively open in X and $X = A_1 \cup A_2$.
- X is called *connected* if it is not disconnected.

3. Evans, p. 85 problem 3.

4. Evans, p. 86 problem 4.