Partial Differential Equations

Midterm Exam

April 11, 2007

1. (a) Solve the partial differential equation

$$u_t + x^2 \, u_x = 0 \,,$$

where $u = u(x,t)$ on the open first quadrant of the (x,t) plane.
Hint: Show that $z(s) = u(x(s), t+s)$ is constant if $x'(s) = x^2(s)$.

(b) Draw the characteristic curves, then state a set of boundary and/or initial conditions that specify the solution uniquely in the first quadrant of the (x,t) plane.

(10+10)

2. Let $U \subset \mathbb{R}^n$ open and bounded and $u \in C^2(\bar{U})$. We say that u is *superharmonic* if

$$-\Delta u \geq 0 \quad \text{in } U \,.$$

(a) Prove that for superharmonic u,

$$u(x) \geq ⨍_{B(x,r)} u(y) \, dy$$

for all $B(x,r) \subset U$.
Note: Do the computation explicitly. Simply referring to the homework is not enough.

(b) Show that if u is superharmonic and $u \geq 0$ on ∂U, then $u \geq 0$ in $\bar{U}$.

(10+10)

3. Recall that the solution to the heat equation

$$u_t - \Delta u = 0 \quad \text{in } \mathbb{R} \times (0,\infty) \,,$$
$$u = g \quad \text{on } \mathbb{R} \times \{t = 0\}$$

is given by

$$u(x,t) = \int_{\mathbb{R}} \Phi(x-y,t) \, g(y) \, dy \,,$$

where, for $t > 0$,

$$\Phi(z,t) = \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{|z|^2}{4t}}\,.$$

(a) Show that if g is an even function, i.e. if $g(x) = g(-x)$, then $u(\cdot, t)$ is even for every $t \geq 0$.

(b) What does this imply for the solution of the heat equation on the halfline with Neumann boundary conditions, i.e.

$$\begin{aligned} u_t - \Delta u = 0 \quad & \text{in } (0,\infty) \times (0,\infty)\,, \\ u_x = 0 \quad & \text{on } \{x = 0\} \times (0,\infty)\,, \\ u = g \quad & \text{on } (0,\infty) \times \{t = 0\}\,? \end{aligned}$$

(10+10)

4. Let $U \subset \mathbb{R}^n$ open and bounded. Recall that for $1 \leq p < \infty$, the L^p-norm of a suitably integrable function u is defined

$$\|u\|_{L^p}^p = \int_U |u(x)|^p \, dx$$

while the L^∞-norm is given by

$$\|u\|_{L^\infty} = \operatorname*{ess\,sup}_{x \in U} |u(x)|\,.$$

(a) Show that if $u \in C_1^2(\bar{U} \times [0,\infty))$ satisfies the heat equation

$$\begin{aligned} u_t - \Delta u = 0 \quad & \text{in } U \times (0,\infty)\,, \\ u = g \quad & \text{on } U \times \{t = 0\}\,, \\ \nu \cdot Du = 0 \quad & \text{on } \partial U \times (0,\infty)\,, \end{aligned}$$

then

$$\|u(\cdot,t)\|_{L^p} \leq \|g\|_{L^p}\,,$$

for $p \geq 2$.

Note: For simplicity, you may assume that p is an even integer.

(b) Prove that, provided $u \in C(\bar{U})$,

$$\lim_{p \to \infty} \|u\|_{L^p} = \|u\|_{L^\infty}\,.$$

Note: The statement is also true if u is only L^∞, but the proof is not so elementary.

(c) Explain why (a) and (b) imply yet another proof of the maximum principle for the heat equation.

(10+10+10)

5. Show that for solutions of the wave equation

$$u_{tt} - \Delta u = 0$$

on $\mathbb{R}^n \times (0, \infty)$, where $u(\,\cdot\,, 0)$ has compact support, the energy

$$E(t) = \int_{\mathbb{R}^n} \left(u_t^2 + |Du|^2\right) dx$$

is constant in time. (10)