General Mathematics and Computational Science II

Final Exam

May 26, 2007

Notation

- For $v \in \mathbb{C}^N$, the discrete Fourier transform of v is defined

$$\hat{v}_k = \frac{1}{N} \sum_{j=0}^{N-1} \mathrm{e}^{-\mathrm{i}jkh} \, v_j$$

  with $h = 2\pi/N$ and for $k = 0, \dots, N-1$.

- For G a finite abelian group, written additively and $f \colon G \to \mathbb{C}$, the Fourier transform of f is defined

$$\hat{f}(\chi) = \sum_{a \in G} \chi(a) \, f(a)$$

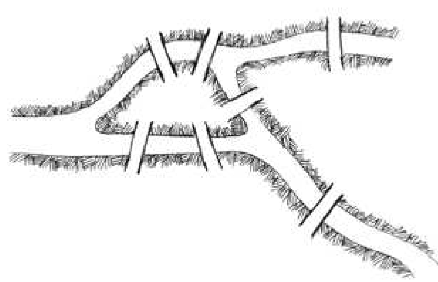
  for $\chi \in \hat{G}$.

1. Let H_P denote the central reflection about a point P of the plane.

   Show that $H_A H_B = H_B H_C$ if and only if B is the midpoint of the line segment AC.

   (10)

2. The concept of Eulerian graphs goes back to the famous "Königsberg bridge problem": Is it possible to take a walk of Königsberg such that every bridge in the map below is crossed exactly once? Answer this question giving a concise proof.



(Figure from `http://bridges.canterbury.ac.nz/features/bridges.html`)

(10)

3. Solve the linear programming problem

$$\text{minimize } z = 5\,x + 2\,y$$

subject to

$$\begin{aligned} 2\,x + 3\,y &\geq 6\,, \\ 4\,x + y &\geq 6\,, \\ x &\geq 0\,, \\ y &\geq 0\,, \end{aligned}$$

using either the graphical method or the simplex method. (10)

4. Show that the leaving variable in one iteration of the simplex method can never be entering variable in the next iteration. (10)

5. Let M be a finite set and let $\mathrm{Sym}(M)$ denote the permutations of M, i.e., the set of bijective maps from M to itself.

   (a) Show that $\mathrm{Sym}(M)$ is a group whose group operation is the composition of maps.

   (b) What is the order, i.e., the number of elements of $\mathrm{Sym}(M)$?

   (c) Is $\mathrm{Sym}(M)$ abelian? Give a proof or state a counter-example.

(10+5+5)

6. Let G be a finite group, and let $g \in G$.

   (a) Show that there exists an $n \in \mathbb{N}$ such that $g^n = e$, the identity in G.

   (b) The abstract statement of part (a) actually proves something about the Kac ring model. What is it? Explain the correspondence concisely.

(8+7)

7. Let G be a finite abelian group, $\psi \in \hat{G}$ a character, and $f \colon G \to \mathbb{C}$ a function on G. Set $g = \psi f$.

   Find an expression for the Fourier transform of g in terms of the Fourier transform of f. (10)

8. Let

$$u_j = \begin{cases} 1 & \text{if } j \text{ is even} \\ -1 & \text{if } j \text{ is odd} \end{cases}$$

and suppose that N is even. Show that its discrete Fourier transform satisfies

$$\hat{u}_k = \begin{cases} 1 & \text{if } k = N/2 + mN \text{ for some } m \in \mathbb{Z} \\ 0 & \text{otherwise}\,. \end{cases} \tag{15}$$