General Mathematics and Computational Science II

Exercise 18

April 26, 2007

Recall that the discrete Fourier transform of an N-tuple of numbers $u_1, \ldots, u_N$ is defined

$$\hat{u}_k = \frac{1}{N} \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}kx_j}\, u_j$$

for $k = 1, \ldots, N$, where $x_j = jh$ with $h = 2\pi/N$.

1. Prove *Parseval's identity*

$$\sum_{k=1}^{N} |\hat{u}_k|^2 = \frac{1}{N} \sum_{j=1}^{N} |u_j|^2 \,.$$

2. Show that $\hat{u}_{k+N} = \hat{u}_k$. Vice versa, setting $u_{j+N} = u_j$, show that

$$\hat{u}_k = \frac{1}{N} \sum_{j=m+1}^{m+N} \mathrm{e}^{-\mathrm{i}kx_j}\, u_j$$

   for every integer m.

3. Let

$$w_j = \sum_{l=1}^{N} u_l \, v_{j-l}$$

   with the understanding that the three N-tuples are periodically extended beyond their basic range of definition on which the index varies from 1 to N; cf. Question 2.

   Show that

$$\hat{w}_k = \hat{u}_k \, \hat{v}_k \,.$$