

General Mathematics and Computational Science II

Exercise 5

February 15, 2007

Ehrenfest's urn model (paraphrases from Dorfman's book). Consider two urns, I and II, and a bag. N balls, labeled $1, \dots, N$, are distributed between the urns. The bag contains N pieces of paper that carry numbers $1, \dots, N$. At each time step, somebody draws a number out of the bag at random, moves the corresponding ball from its current to the other urn, and then puts the piece of paper back into the bag. This is repeated a large number of times. Consider the difference

$$D(t) = |N_{\text{I}}(t) - N_{\text{II}}(t)|$$

between the number of balls in urns I and II as a function of time.

1. In which sense is the behavior of this system time-reversible? Does it have recurrence?
2. Define an entropy function for this system.
3. Find an equation for $\langle D(t) \rangle$. What happens for large t ?

(Notice that the chance that a ball will move from the fuller urn to the emptier urn is always greater than the chance that the opposite will happen, therefore the system has the tendency to approach an equilibrium where the balls are distributed equally between the urns. Fluctuations will also occur, even very large ones.)